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THE MATHEMATICAL GAZETTE

EDITED BY
THE LATE W. J. GREENSTREET, M.A., AND PROF. E. H. NEVILLE, M.A.
WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., F.R.S.
AND
PROF. E. T. WHITTAKER, Sc.D., F.R.S.

LONDON
G. BELL & SONS, LTD., PORTUGAL STREET, KINGSWAY, W.C. 2.
AND BOMBAY

Vol. XV., No. 210. DECEMBER, 1930. 3s. Net.

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The Mathematical Association.

THE ANNUAL MEETING will be held at the LONDON DAY TRAINING COLLEGE, Southampton Row, London, W.C. 1, on *Monday, 5th January*, 1931, at 2.45 p.m., and *Tuesday, 6th January*, 1931, at 10.0 a.m. and 2.30 p.m. The President will deliver his Address on the first day.

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AND BOMBAY.

VOL. XV. DECEMBER, 1930. No. 210.

THE EDITORSHIP OF THE *GAZETTE*.

THE Council of the Mathematical Association has invited Mr. T. A. A. BROADBENT to edit the *Gazette*, and his responsibility will begin with the next issue. Correspondence should be addressed to

2 Buxton Avenue, Caversham, Reading.

THE CIRCUMRADIUS OF THE GENERAL SIMPLEX.

BY H. S. M. COXETER, B.A.

·1. In $n+1$ dimensions, let

$$X_r \quad (r=0, 1, \dots n)$$

be $n+1$ points determining a prime (or n-space), and let O, O' be two points outside this prime. Let $\mathbf{w}_r$, $\mathbf{w}_r'$ denote the vectors $\mathbf{OX}_r$, $\mathbf{O'X}_r$ respectively, and let $\mathbf{OP}$, $\mathbf{O'P'}$ represent the vector sums

$$\sum_0^n x_r\mathbf{w}_r, \quad \sum_0^n x_r\mathbf{w}_r'.$$

Now,
$$\begin{aligned}\mathbf{PP'} &= \mathbf{PO}+\mathbf{OO'}+\mathbf{O'P'}\\ &= \mathbf{OO'}-(\mathbf{OP}-\mathbf{O'P'})\\ &= \mathbf{OO'}-\sum_0^n x_r(\mathbf{w}_r-\mathbf{w}_r')\\ &= \mathbf{OO'}-\sum_0^n x_r\mathbf{OO'}\\ &= (1-\textstyle\sum_0^n x_r)\mathbf{OO'}.\end{aligned}$$

So, if

·11 $$\sum_0^n x_r=1,$$

P' coincides with P, whatever be the position of O'. We may therefore take O' to be the image of O by reflection in the prime $X_0 \dots X_n$, in which case P' is the image of P. Hence P, coinciding with its image, must lie in the prime. Conversely, if P lies in the prime, ·11 follows (O being outside the prime).

The numbers x_r, satisfying ·11, are called *barycentric coordinates* of P, referred to the simplex $X_0 \dots X_n$.

·2. Let x_r+y_r $(r=0, 1, \dots n)$, satisfying

·21 $$\sum_0^n (x_r+y_r)=1,$$

be the barycentric coordinates of another point Q (in the n-space $X_0 \dots X_n$). Then

$$\mathbf{PQ}=\mathbf{OQ}-\mathbf{OP}$$

$$=\sum_0^n (x_r+y_r)\mathbf{w}_r-\sum_0^n x_r\mathbf{w}_r$$

$$=\sum_0^n y_r\mathbf{w}_r.$$

Taking scalar squares,

$$PQ^2=\sum_0^n\sum_0^n y_r y_s(\mathbf{w}_r\mathbf{w}_s).$$

Now, if l_{rs} denote the length of the edge X_rX_s of the simplex of reference, with the convention

·22 $$l_{sr}=-l_{rs},$$

implying

·23 $$l_{rr}=0,$$

we have $$l_{rs}^2=(\mathbf{w}_s-\mathbf{w}_r)^2=\mathbf{w}_s^2+\mathbf{w}_r^2-2(\mathbf{w}_r\mathbf{w}_s),$$

and therefore

$$\sum_0^n\sum_0^n y_r y_s l_{rs}^2=\sum_0^n\sum_0^n y_r y_s\mathbf{w}_s^2+\sum_0^n\sum_0^n y_r y_s\mathbf{w}_r^2-2\sum_0^n\sum_0^n y_r y_s(\mathbf{w}_r\mathbf{w}_s)$$

$$=2\sum_0^n y_r\cdot\sum_0^n y_s\mathbf{w}_s^2-2PQ^2.$$

But, by ·11 and ·21,

·24 $$\sum_0^n y_r=0.$$

Hence

·25 $$-2PQ^2=\sum_0^n\sum_0^n y_r y_s l_{rs}^2.$$

We have thus expressed the distance between two points in terms of the differences of their barycentric coordinates.*

·3. Let $(x_0, \dots x_n)$ be the "circumcentre" of the simplex $X_0 \dots X_n$; i.e. the point whose distance, R, from X_q, is independent of q. In this case, the appropriate y's are

$$y_r=x_r \ (r\neq q), \ y_q=x_q-1.$$

Hence, by ·25,

$$-2R^2=\sum_0^n\sum_0^n y_r y_s l_{rs}^2$$

·31 $$=\sum_0^n\sum_0^n x_r x_s l_{rs}^2-2\sum_{s=0}^n x_s l_{qs}^2.$$

It follows that $\sum_{s=0}^n x_s l_{qs}^2$ is independent of q.

---

* The y's, satisfying ·24, may be regarded as coordinates of the "point at infinity" on the line PQ. Two lines PQ, $P'Q'$ are perpendicular if the scalar product $(\mathbf{PQ}.\mathbf{P'Q'})$ vanishes. Work very similar to the above gives the condition

$$\sum_0^n\sum_0^n y_r y_s' l_{rs}^2=0.$$

Also

$$\sum_0^n \sum_0^n x_r x_s l_{rs}^2 = \sum_0^n x_r \sum_{s=0}^n x_s l_{rs}^2 = \sum_0^n x_r \sum_{s=0}^n x_s l_{qs}^2 = \sum_{s=0}^n x_s l_{qs}^2.$$

Therefore, by ·31,

·32 $$\sum_{s=0}^n x_s l_{qs}^2 = 2R^2. \quad (q=0, 1, \dots n)$$

·11 and ·32 provide $n+2$ linear equations for R^2 and the x's. Eliminating the latter, we obtain

33 $$\begin{vmatrix} 1 & 1 & 1 & 1 & \dots \\ 2R^2 & & & & \\ 1 & l_{00}^2 & l_{01}^2 & l_{02}^2 & \dots \\ 1 & l_{10}^2 & l_{11}^2 & l_{12}^2 & \\ 1 & l_{20}^2 & l_{21}^2 & \ddots & \\ \vdots & \vdots & & l_{nn}^2 & \end{vmatrix} = 0,$$

whence

·34 $$-2R^2 = \begin{vmatrix} l_{00}^2 & l_{01}^2 & l_{02}^2 \dots \\ l_{10}^2 & l_{11}^2 & l_{12}^2 \\ l_{20}^2 & l_{21}^2 & \ddots \\ \vdots & & l_{nn}^2 \end{vmatrix} \div \begin{vmatrix} 0 & 1 & 1 & 1 & \dots \\ 1 & l_{00}^2 & l_{01}^2 & l_{02}^2 & \dots \\ 1 & l_{10}^2 & l_{11}^2 & l_{12}^2 & \\ 1 & l_{20}^2 & l_{21}^2 & \ddots & \\ \vdots & \vdots & & l_{nn}^2 & \end{vmatrix}.$$

(Of course, $l_{sr}^2 = l_{rs}^2$ and $l_{rr}^2 = 0$.)

·4. This result can alternatively be deduced from formula (5·1) of D. M. Y. Sommerville's *Geometry of N Dimensions* (1929; p. 125).

For, if X_{n+1} is the circumcentre of the simplex $X_0 \dots X_n$, $X_0 \dots X_{n+1}$ is an $(n+1)$-dimensional simplex of zero content. So, by the above-mentioned formula,

$$0 = \begin{vmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & l_{00}^2 & l_{01}^2 & \dots & l_{0n}^2 & R^2 \\ 1 & l_{10}^2 & l_{11}^2 & \dots & l_{1n}^2 & R^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & l_{n0}^2 & l_{n1}^2 & \dots & l_{nn}^2 & R^2 \\ 1 & R^2 & R^2 & \dots & R^2 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & l_{00}^2 & l_{01}^2 & \dots & l_{0n}^2 & 0 \\ 1 & l_{10}^2 & l_{11}^2 & \dots & l_{1n}^2 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & l_{n0}^2 & l_{n1}^2 & \dots & l_{nn}^2 & 0 \\ 1 & 0 & 0 & \dots & 0 & -2R^2 \end{vmatrix}$$

$$= -2R^2 \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & l_{00}^2 & l_{01}^2 & \dots & l_{0n}^2 \\ 1 & l_{10}^2 & l_{11}^2 & \dots & l_{1n}^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & l_{n0}^2 & l_{n1}^2 & \dots & l_{nn}^2 \end{vmatrix} - \begin{vmatrix} l_{00}^2 & l_{01}^2 & \dots & l_{0n}^2 \\ l_{10}^2 & l_{11}^2 & \dots & l_{1n}^2 \\ \cdot & \cdot & \cdot & \cdot \\ l_{n0}^2 & l_{n1}^2 & \dots & l_{nn}^2 \end{vmatrix}.$$

H. S. M. COXETER.

MATHEMATICS FOR THE STUDY OF FREQUENCY STATISTICS.

By W. F. Sheppard, Sc.D., LL.M.

(*Presidential Address to the Mathematical Association*, 1930.)

1. The title of my address is "Mathematics for the Study of Frequency Statistics."

By frequency statistics I mean statistics which give the result of counting. A statement as to the number of teachers receiving annual salaries in excess of a certain amount would come under the head of frequency statistics: a statement as to the total amount paid annually to these teachers would not. The sort of thing that I have mainly in mind is a table giving the numbers of boys of specified heights, the heights being taken to the nearest inch, or the numbers of girls who, having worked a paper with full marks 100, get under 5 marks, 5 and under 10, and so on. I may mention that the word "frequency" is usually applied to the actual number in a specified category; the proportion of this number to the total is called the "relative frequency."

2. *Possibilities.* I am not proposing to give a general sketch of the mathematical theory of statistics. Some of the mathematics is difficult, and it is of a rather special kind. And we should not present it cut and dried to our scholars. I suggest that, ordinarily, the collection of statistics should come first, their study next, and the mathematics last, when the pupil begins to see what sort of thing is needed.

It should be remembered, also, that there is a very close connexion between the study of statistics and the theory of probability. It is, indeed, rather difficult to separate the two, and they might even be regarded as different aspects of the same subject. Some of you may remember a paper by Mr. Hope-Jones, published in the *Mathematical Gazette* of July, 1924, "A Plea for Teaching Probability in Schools." He shows that the subject can be made very interesting, without bringing in too abstruse mathematics.

3. *Some aspects.* Here are some points for consideration, in reference to the desirability of finding a place for the study of statistics in schools.

(i) Statistical facts do come into the curriculum, if only as exercises in arithmetic.

(ii) These facts give opportunity for graphical representation, and this leads to a more exact and more scientific handling of graphs.

(iii) When we get to mathematical treatment, the subject gives additional material for exercises in branches of mathematics whose practical utility is not always clear even to the higher classes in schools: in differential and integral calculus, for instance.

(iv) Finally, study of a particular subject, whether in school or later on, may lead to special attention being paid to allied branches of mathematics, and even to new developments in these branches. Attempts at scientific gambling led to the theory of permutations and combinations, and thence to various matters of interest, such as Bernoulli's formula, which I shall mention later; the study of interpolation led to the binomial theorem; and so on.

It is mainly with this latter aspect that I am dealing to-day. I will run through the general theory briefly, mentioning some special matters for mathematical treatment: and I will then deal in more detail with two branches of mathematical work which are of special interest, namely (1) central differences and sums, and (2) the Euler-Maclaurin formula and allied theorems.

4. *General theory.* Coming now to the general theory, I want to point out that the school study of statistics differs from purely mathematical work in that we begin with observed facts rather than with theoretical processes. The theory of statistics thus forms part of the theory of observations, and their

mathematical treatment forms part of what Professor Whittaker has very happily called the *Calculus of Observations.** It is therefore very different from such a subject as coordinate geometry—which is often confused with the theory of graphs—and there are two particular points in which the difference is striking.

5. *Statistics and pure mathematics contrasted.* Suppose that in our statistics we are dealing with a single variable, say height. Call this variable V. Then we take a series of values of V as determining a series of classes, and our "entries," to use a technical term relating to mathematical tables, are the numbers of individuals falling into these classes. We might, for instance, tabulate by intervals of 2 inches, so that the entry m is the number of individuals for whom V is between $a-1$ in. and $a+1$ in.

Now compare a table of this kind, giving m in terms of V, with the equation to a circle, which gives y in terms of x. Then we have these two contrasts.

(i) First, in the case of the circle, we are able to choose any value of the abscissa x (within certain limits), and calculate y accordingly. But, in the case of observations, the values of m are given only for particular values of V. If we wanted to subdivide the classes—for instance, to deduce a table with difference 1 inch instead of 2 inches—we should have to interpolate. Here, therefore, interpolation comes in. I do not say that actual interpolation is often required in dealing with statistics of the particular kind that I have mentioned. But the interpolation formula is a thing that we ought to have in mind, because it can be used for other purposes, such as expressing derivatives in terms of the data. Here central-difference formulae are important; and I propose to say something presently about central differences and sums (see Excursus I = §§ 11-24).

(ii) The other point in which the treatment of observational data differs from the treatment of the equation to a circle is this. In the case of an equation, say $x^2+y^2=10$, we can not only, as I have said, choose any value of x that we like, say $x=2$: but we can then calculate y to as great a degree of accuracy as we please. But in the other case our data, being the result of observation, are only as exact as our powers of observation can make them. They are liable to error: and this liability to error affects not only the data but also the mathematical processes, such as interpolation, to which I have already referred.

6. *Accuracy of numerical data.* On the second of these two points there is a possible criticism that may be made. Our observations show that in a particular class there is a certain number of individuals, say m. Surely, you will say, there is no possibility of error here. We want exactness: and the only possible exactness, in this world, is exactness of number.

This is quite true. But what we are considering now is not really an error in observation of the number in a particular class, but a deviation of this number from an unknown norm: the difference between what ought to be and what is. And the difficulty of the subject—the subject known as the theory of error—is that we do not know the "what ought to be."

7. *Basis of theory of error.* You may throw a die 1200 times. The number of times a 6 is thrown will be somewhere near 200. Suppose it is 210. Have another 1200 throws. Then you may get a different number of 6's. Go on long enough, and you will come to the conclusion that there is a number u, not necessarily an integer, which is, on the average, the most probable number of times, in 1200, that the throw will be 6. But we can never know u exactly. And yet we can deal with the data on the assumption that there is a u; we can find approximately the probability that it lies within any stated limits; and so on.

* E. T. Whittaker and G. Robinson, *The Calculus of Observations* (1st ed. 1924).

This, however, comes under the head of "inverse probability": a difficult subject. The accepted method of treatment is to begin with some study of direct probability. We draw a card at random from a pack. Suppose the probability of drawing a court card is p, and that of not drawing a court card is $q \equiv 1 - p$. Then, adopting certain axioms, the probability that in n drawings we should draw a court card m times is $(n, m)p^m q^{n-m}$, where (n, m) is the binomial coefficient. This is easily calculated. But suppose that n is large, and that we want to know the probability that the number of times a court card is drawn will be between m_1 and $m_1 + 100$. This would mean calculating the sum of 100 arithmetical terms: which is troublesome. An approximate value for the sum is given by the Gaussian formula, represented graphically by means of the "normal figure of frequency." This is the familiar "cocked-hat" figure, shown in Fig. 1. OA is the central ordinate, about which the

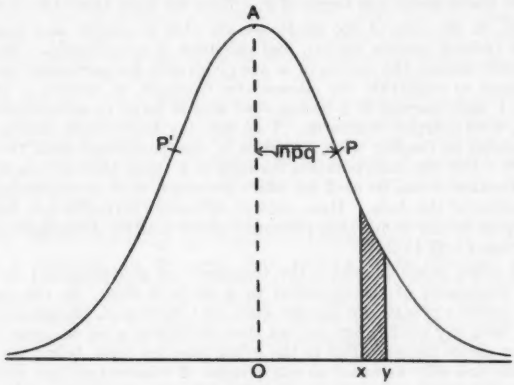


FIG. 1.

figure is symmetrical. There are points of inflexion at P and P'; and the distance of P or P' from OA represents the "standard deviation", which is $\sqrt{(npq)}$. The most probable drawing is (nearly) np court cards and nq others. We want to know the probability that the number of court cards will be between $np + x$ and $np + y$. In the figure, O corresponds to the drawing of np court cards. We find the ordinates whose distances from OA represent x and y: and the probability we are seeking is the ratio of the area between these two ordinates to the total area of the figure.

8. *Gaussian distributions.* The special importance of the Gaussian figure lies in the fact that it applies not only to frequency of the errors I have just been considering—called errors of random sampling—but also roughly to variations from a mean in cases of various kinds. Quetelet, the Belgian astronomer, called attention to several: temperatures, prices of grain, astronomical observations, heights and chest-measurements of men. The resemblance was so striking as almost to suggest that nature was striving to attain to a standard value, and that the deviations from this value were really "errors," in the ordinary sense of the word.

9. *Constants of a distribution.* When we have reason to suppose that our data represent the result of random sampling from classes whose numbers are related in a particular way but involve unknown constants, the facts may be concisely represented by the best values obtainable for these constants. The

meaning of "best," and the methods of obtaining the best values, constitute a large branch of the subject. Here I need only mention that in certain kinds of cases the best—or nearly best—values are to be found by the method of moments, that is, by equating the first two or three moments of the data to those of the hypothetical distribution. Here we come in touch with quadrature, where the Euler-Maclaurin formula is of special importance. This is the second matter with which I shall deal. (See Excursus II = §§ 25-36.)

We must not, however, attach too much importance to the use of moments for the determination of unknown constants. The methods used are not always sound, and the sound methods are troublesome.

10. *Correlation.* I come finally to correlation. The modern mathematical treatment of this subject was initiated by Francis Galton, and was carried on by Edgeworth, Karl Pearson, and others. The special aspect to which Galton called attention was *regression* of successive generations towards the mean. The sons of tall men are—on the average—tall, but not so tall: the sons of short men are short, but not so short. But this does not mean that the parents of tall men are taller, or the parents of short men shorter. The regression works both ways.

We find the same kind of relation between measurements in individuals: measurements, for instance, of height and span. Where height is above the average, span is also, on the whole, above the average; and so on. We deal with the data mathematically—on the assumption, in the simpler cases, of normal variation and "normal correlation"—by getting an expression for the frequency of joint occurrence of deviations, to a stated extent, from the respective means. But these deviations need not be the deviations of two measures only: there may be a large number of measures concerned.

We have to deal with still larger numbers of variables when we are considering the correlation of errors in a statistical table.

There is therefore a great complexity in the mathematical work in relation to questions of this kind. The complexity can be considerably reduced by using the notation of the tensor calculus.

Excursus I. Central Differences and Sums.

(*A*) *Central-Difference Formulae.*

11. *Interpolation by first difference.* You are all familiar with ordinary first-difference interpolation—what used to be called the theory of proportional parts. It occurs in the use of a mathematical table. If, for instance, your table gives you

No.	Log.
2·16	·3345
2·17	·3365

and if you are satisfied that between 2·16 and 2·17 the logarithm of x increases uniformly as x increases, then you can say at once that the logarithm of 2·164 is $\cdot3345 + \frac{4}{10}$ of $\cdot0020 = \cdot3353$. Graphically, we take the upper boundary of the graph of $\log x$ with regard to x to be a straight line.

Analytically, we can take a larger part of the table, and regard it as a whole. Taking x to be the number, and u its logarithm, we have a series of values of x, proceeding by a common difference h, each with its logarithm. Denoting the values of x by $x_0, x_1, x_2 \ldots$, the corresponding u's are $u_0, u_1, u_2, \ldots$, as shown in Table I. We want to find the value of $\log(x_0 + \theta h)$, where θ is numerical and is usually between 0 and 1. This value, on the same system, is denoted by u_θ.

Table 1.

x	u
x_0	u_0
$x_0+h=x_1$	u_1
$x_1+h=x_2$	u_2

12. *Various forms.* Although the expression for u_θ is very simple, it can take a good many different forms. These different forms correspond usually to the beginnings of different formulae for less simple cases, and it seems worth while to set them out here. I will give each expression, with the name of the corresponding more general formula.

(i) We can use the 1st difference Δ. This, as we can see arithmetically or graphically (Fig. 2), gives

$$u_\theta = u_0 + \theta\Delta. \qquad \text{.....}(GN)$$

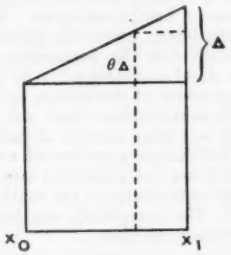


FIG. 2.

I mark this *GN*, because it is the basis of formulae of the Newtonian type, beginning with Gregory's formula (1670*), which I will mention presently.

(ii) A different method, which corresponds to a method adopted for more general formulae, is to write

$$u_\theta = A + B\theta,$$

and find A and B from the fact that u_θ is u_0 for $\theta=0$ and u_1 for $\theta=1$. This gives

$$u_\theta = u_0 + (u_1 - u_0)\,\theta.$$

This, of course, is exactly the same thing as the *GN* form, but it leads to the alternative form

$$u_\theta = \theta u_1 + (1-\theta)\,u_0, \qquad \text{.....}(L).$$

where u_θ is expressed in terms of the u's. I have marked this L, because it is the simplest case of the Lagrangian type of formula, in which u_θ is expressed in terms of the u's, whereas in the Gregory-Newton type it is expressed in terms of differences of the u's.

(iii) There is one more form. If we write

$$\phi \equiv 1 - \theta,$$

so that $\theta+\phi=1$, then L takes the form

$$u_\theta = \theta u_1 + \phi u_0, \qquad \text{.....}(E)$$

which is the basis of Everett's formula.

* Dates, throughout, are mostly taken from Whittaker and Robinson's book, quoted in note to § 4.

(iv) The formula

$$u_\theta = A + B\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (T)$$

used in (ii) is really the elementary form of Taylor's theorem. This theorem is very important in connexion with interpolation by means of differences.

13. *Inspection of differences.* All this is on the supposition that 1st differences of the u's are practically constant, so that we need only use two of the u's. But in actual practice, apart from mathematical tables, we do not get these simple cases. We do, however, have cases in which u_θ can, in the neighbourhood of the particular value with which we are concerned, be represented approximately by a polynomial function of θ. The formulae we are considering can be extended to meet such cases; the principle of the extension being that the pth difference of a polynomial function of θ of degree p is constant, and the $(p+1)$th difference is zero. We can therefore find the $p+1$ constants in the function by equating the values of $p+1$ consecutive u's to the corresponding values of the function. Different ways of choosing these u's, in relation to u_θ, lead to different formulae.

For illustration, take the case of a table of tangents to 3 decimal places, the values of x proceeding by difference 1/6th of a degree. Part of the table is given in Table 2.

Table 2.

x	u	1st diff.
⋮	⋮	+
		⋮
70° 0′	2·747	
		26
10′	2·773	
		25
20′	2·798	
		26
30′	2·824	
⋮		⋮

Here 1st-difference interpolation will give a good result. But this is not the case when we get near 90°, as is seen from Table 3.

Table 3.

x	u	1st diff.	2nd diff.	3rd diff.	4th diff.
⋮	⋮	+	+	+	⋮
86° 00′	14·301	⋮	⋮	⋮	⋮
		623		⋮	⋮
10′	14·924		58		⋮
		681		6	
20′	15·605		64		+4
		745		10	
30′	16·350		74		+3
		819		13	
40′	17·169		87		0
		906		13	
50′	18·075		100		+6
		1006		19	
87° 00′	19·081		119	⋮	⋮
		1125	⋮		
10′	20·206	⋮			
⋮	⋮				

Here 4th differences are small, but not absolutely negligible; differences after the 4th may be treated as negligible. We ought therefore to take u_θ to be equal to a polynomial of the 4th degree in θ.

14. *Interpolation with higher differences.* Let us suppose, then, that the data we are dealing with are of this latter kind; *i.e.* that differences up to the 4th must be taken into account, but that differences after the 4th are negligible.

For 1st-difference interpolation I called the x's x_0 and x_1, and the u's u_0 and u_1. But there is no special virtue in the suffixes 0 and 1; indeed, they cause a difficulty by seeming to imply that we do not know any values before u_0. I will therefore take the given x's to be x_{15}, x_{16} ... x_{22}, and the corresponding u's to be u_{15}, u_{16} ... u_{22}. And I will suppose that we want to find the value of $u_{18+\theta}$, where, for precision, I will take θ to be between 0 and $\frac{1}{2}$.

Our data will then be a table of the form of Table 4.

Table 4.

x	u	1st diff.	2nd diff.	3rd diff.	4th diff.
x_{15}	u_{15}				
		Δu_{15}			
x_{16}	u_{16}		$\Delta^2 u_{15}$		
		Δu_{16}		$\Delta^3 u_{15}$	
x_{17}	u_{17}		$\Delta^2 u_{16}$		$\Delta^4 u_{15}$
		Δu_{17}		$\Delta^3 u_{16}$	
x_{18}	u_{18}		$\Delta^2 u_{17}$		$\Delta^4 u_{16}$
		Δu_{18}		$\Delta^3 u_{17}$	
x_{19}	u_{19}		$\Delta^2 u_{18}$		$\Delta^4 u_{17}$
		Δu_{19}		$\Delta^3 u_{18}$	
x_{20}	u_{20}		$\Delta^2 u_{19}$		$\Delta^4 u_{18}$
		Δu_{20}		$\Delta^3 u_{19}$	
x_{21}	u_{21}		$\Delta^2 u_{20}$		
		Δu_{21}			
x_{22}	u_{22}				

The formulae which are applicable to such data—restricting ourselves to the polynomial method—are of three main types, of which the first is the best known but the third is possibly the most useful. Before considering these formulae, there are some brief observations to be made on the above notation.

15. *Ordinary notation.* The notation used in Table 4 explains itself, but the following points may be noted.

In dealing with 1st-difference interpolation I denoted $u_1 - u_0$ by Δ. This was good enough for our purpose; but we now need more precision. We therefore, for all the differences, use suffixes corresponding to the suffixes of the u's. The established practice is that Δu_{15} means $u_{16} - u_{15}$, and that we call this the 1st difference of u_{15}. Similarly $\Delta^2 u_{15}$ really means $\Delta\Delta u_{15}$, *i.e.* the 1st difference of Δu_{15}, so that

$$\Delta^2 u_{15} = \Delta(\Delta u_{15}) = \Delta u_{16} - \Delta u_{15} = u_{17} - 2u_{16} + u_{15}.$$

These are called *advancing differences.* But the notation is quite arbitrary. We might just as well have used Δu_{15} to mean $u_{15} - u_{16}$ or $u_{14} - u_{15}$ or $u_{15} - u_{14}$; and in fact special symbols are sometimes used for differences at the end of the range.

Also I suggest that it is not desirable to introduce the idea of operators too early. We should not define Δu_{20} as the result of the operation denoted by Δ acting on u_{20}. We have a series of u's, and a series of differences; and we give suffixes to the differences to show where they are to be placed in the table. Separation of symbols of operation comes later.

It should be noticed that $\Delta^m u_{15}$ can be expressed in terms of $u_{15}, u_{16} \ldots u_{15+m}$.

16. *First type of formula.* We return now to the types of formulae, mentioned in § 14.

The best-known formula of the first type is the *Gregory-Newton* formula :

$$u_{18+\theta} = u_{18} + \theta\Delta u_{18} + \frac{\theta(\theta-1)}{2!}\Delta^2 u_{18} + \ldots + \frac{\theta(\theta-1)(\theta-2)(\theta-3)}{4!}\Delta^4 u_{18}.$$

This was discovered independently by James Gregory (1670) and by Newton (1687). James Gregory seems to have been a rather remarkable man. He was at an early age the inventor, or one of the inventors, of the reflecting telescope, and was a brilliant but contentious mathematician. We know him by name from " Gregory's series " in trigonometry. He died at the age of thirty-six, only about a year after he had been appointed professor of mathematics at Edinburgh.

This formula is the *advancing-difference* formula. By converting differences into the corresponding u's (§ 15), it will be seen that it involves the five u's from u_{18} to u_{22}.

There is a corresponding *receding-difference* formula, which contains u_{19} and the differences in a diagonal line backwards to $\Delta^4 u_{15}$. The u's involved in this formula are from u_{19} to u_{15}. The formula is easily obtained by inverting the table.

17. *Central-difference formulae.* Both the above formulae—the advancing-difference and the receding-difference—are rather lop-sided. We are finding a value between u_{18} and u_{19}. Each of the formulae begins by taking account of the pair of u's ; but each then goes off on a diagonal line away from the pair. If we want as accurate a result as possible, it would seem more natural to keep as close to the pair as possible. Newton, followed by other writers, devised formulae on this principle.

(*a*) The *Newton-Gauss* formula, for any value of x between x_{18} and x_{19}, takes terms in a zig-zag horizontal band through u_{18} and Δu_{18} ; *i.e.* it uses u_{18}, Δu_{18}, $\Delta^2 u_{17}$, $\Delta^3 u_{17}$, $\Delta^4 u_{16}$. (I am, unless otherwise stated, considering cases in which we stop at the 4th difference.)

(*b*) The *Newton-Gauss backward* formula is on the same principle, but begins with u_{19} and Δu_{18}, ending with $\Delta^4 u_{17}$.

(*c*) The *Newton-Stirling* formula takes as its first term u_{18}, and takes in pairs the differences of odd order lying immediately above and below the horizontal line through the first term. To keep the formula neat, the mean of each such pair is used. Thus, if we start with u_{18}, we use

$$u_{18},\ \tfrac{1}{2}(\Delta u_{17} + \Delta u_{18}),\ \Delta^2 u_{17},\ \tfrac{1}{2}(\Delta^3 u_{16} + \Delta^3 u_{17}),\ \Delta^4 u_{16}.$$

This formula can equally well be used with negative values of θ, so that the formula which uses u_{18} is convenient for interpolating between $u_{17\frac{1}{2}}$ and $u_{18\frac{1}{2}}$.

(*d*) The *Newton-Bessel* formula is constructed on the same general principle, the difference being that we take the horizontal line through the middle of the interval. Thus, for interpolation between u_{18} and u_{19}, we use

$$\tfrac{1}{2}(u_{18} + u_{19}),\ \Delta u_{18},\ \tfrac{1}{2}(\Delta^2 u_{17} + \Delta^2 u_{18}),$$

and so on.

18. *Central-difference notation.* These formulae of Newton's are better, for certain purposes, than Gregory's advancing-difference formula. But the value of central differences is not limited to formulae of this type, or even to interpolation as a whole (cf. § 5 (*i*)). And to enable central differences—and central sums—to have full play, we must let them have their own notation.

It is easy to see that two things are required. We need a special set of symbols for the differences, with proper suffixes : and we also need a symbol for the process of taking the mean of two consecutive u's or differences.

First, as to differences. I have said (§ 15) that the use of the symbol Δu_{15} to mean $u_{16} - u_{15}$ is quite arbitrary, and that we might just as well have used Δu_{16}. It has become settled practice that Δu_{15} is to mean $u_{16} - u_{15}$; so we

must leave Δ alone and take another letter. For central-difference work we use the small Greek δ; and we denote $u_{16}-u_{15}$ by $\delta u_{15\frac{1}{2}}$, because it lies in the horizontal space between u_{15} and u_{16}. Thus we get a table of differences (Table 5):

Table 5.

x_{17}	u_{17}		$\delta^2 u_{17}$		.....
		$\delta u_{17\frac{1}{2}}$		$\delta^3 u_{17\frac{1}{2}}$	.....
x_{18}	u_{18}		$\delta^2 u_{18}$		.....
		$\delta u_{18\frac{1}{2}}$		$\delta^3 u_{18\frac{1}{2}}$	.....
x_{19}	u_{19}		$\delta^2 u_{19}$		.....
		$\delta u_{19\frac{1}{2}}$		$\delta^3 u_{19\frac{1}{2}}$	.....
x_{20}	u_{20}		$\delta^2 u_{20}$		.....

Next, we use the small Greek μ for the process of taking the mean of a pair of consecutive entries in a column; the mean of the suffixes being taken, as in the case of δ. Thus:

$$\mu\delta^3 u_{18} \equiv \tfrac{1}{2}(\delta^3 u_{17\frac{1}{2}} + \delta^3 u_{18\frac{1}{2}});$$

though, when we have got into the habit of dealing with these symbols, we think of $\mu\delta^3 u_{18}$ as a function of x_{18}, the above formula showing how it is to be calculated.

If we take the means of pairs of entries in this way, and enter them in the table, we get the complete table (Table 6) required for central-difference work:

Table 6.

x_{17}	u_{17}	$(\mu\delta u_{17})$	$\delta^2 u_{17}$	.....
	$(\mu u_{17\frac{1}{2}})$	$\delta u_{17\frac{1}{2}}$	$(\mu\delta^2 u_{17\frac{1}{2}})$	.....
x_{18}	u_{18}	$(\mu\delta u_{18})$	$\delta^2 u_{18}$	.....
	$(\mu u_{18\frac{1}{2}})$	$\delta u_{18\frac{1}{2}}$	$(\mu\delta^2 u_{18\frac{1}{2}})$	.....
x_{19}	u_{19}	$(\mu\delta u_{19})$	$\delta^2 u_{19}$	.....
	$(\mu u_{19\frac{1}{2}})$	$\delta u_{19\frac{1}{2}}$	$(\mu\delta^2 u_{19\frac{1}{2}})$	.....
x_{20}	u_{20}	$(\mu\delta u_{20})$	$\delta^2 u_{20}$	.....

The brackets are entered only in order to show which are the inserted terms. These terms form part of the system: thus the central differences of u_{18} are $\mu\delta u_{18}$, $\delta^2 u_{18}$, $\mu\delta^3 u_{18}$ ..., and those of $\mu u_{18\frac{1}{2}}$ are $\delta u_{18\frac{1}{2}}$, $\mu\delta^2 u_{18\frac{1}{2}}$, ... It will be seen that the terms in any horizontal line all have the same subscript.

Using this notation, we get the various formulae in a different form. For example, the Newton-Stirling formula uses u_{18}, $\mu\delta u_{18}$, $\delta^2 u_{18}$, $\mu\delta^3 u_{18}$ ... .

19. *Everett's formula.* The formulae of the first type include Everett's beautiful formula. If we write

$$\phi \equiv 1-\theta,$$

then we have (reverting to u_0 and u_1),

$$\left.\begin{aligned} u_\theta = \theta u_1 + \frac{\theta(\theta^2-1^2)}{3!}\delta^2 u_1 + \frac{\theta(\theta^2-1^2)(\theta^2-2^2)}{5!}\delta^4 u_1 + \dots \\ + \phi u_0 + \frac{\phi(\phi^2-1^2)}{3!}\delta^2 u_0 + \frac{\phi(\phi^2-1^2)(\phi^2-2^2)}{5!}\delta^4 u_0 + \dots \end{aligned}\right\}.$$

The formula is not very convenient for isolated interpolations, unless θ is a very simple fraction; but it is of great value for construction of tables by subtabulation.*

* Subtabulation is the process of constructing from a table, in which the constant difference of x is h, a new table in which the constant difference is h/c, c being some integer. Usually c is 10; *e.g.* h may be ·01, and h/c is then ·001.

It has been pointed out by Dr. Lidstone* that the formula for u in terms of two consecutive u's and their even central differences was given by Laplace in 1779. But Laplace's formula is not so neatly expressed as Everett's, and it requires at least two changes in order to get it into a form in which it is equally useful. (The formula expresses y_{x+i} in terms of y_x and y_{x-1} and their central differences; and we should have to change i to $\theta - 1$ and x to $X + 1$.)

20. *Second type of formula.* The interpolation formulae that I have so far been considering are those of the first type, involving a tabulated u and a series of differences. By expressing each difference in terms of u's (§ 15) we obtain a formula of the second type, due to Lagrange. The formula is not of importance for our present purpose, except in so far as it demonstrates a rather important property.

Suppose we have to go to 5th differences. Then the advancing-difference formula for finding u_θ, where $0 < \theta < 1$, involves $u_0, u_1 \ldots u_5$. But there is nothing in the formula that limits it to cases in which θ lies between 0 and 1: it could be used for any value of θ, or at any rate for any value that lies between 0 and 5. In particular, the formula, besides being an advancing-difference formula when θ lies between 0 and 1, is a central-difference formula when θ lies between 2 and 3, and a receding-difference formula when it lies between 4 and 5.

For practical use, the Lagrange formula has the disadvantage that it does not show, for any particular case, how many u's are to be taken into account. The differences have to be found; and, having found them, we may as well use them.

21. *Third type of formula.* Finally, so far as interpolation is concerned, we have to consider the third type of formula. In the first type, u_θ was expressed in terms of a u and its differences of successive orders, the coefficients of the terms being expressed in terms of θ. In the second type, u_θ was expressed in terms of a series of u's, the coefficients being expressed in terms of θ. In the third type the successive terms contain successive powers of θ, and the coefficients of these are expressed in terms of a u and its differences. We assume that u_θ is of the form

$$u_\theta = c_0 + c_1\theta + c_2\frac{\theta^2}{2!} + c_3\frac{\theta^3}{3!} + \cdots,$$

and, to complete the formula, we have to find the values of the c's. This is in effect our old friend Taylor's theorem; the difference being that the c's are not usually given explicitly, but have to be determined from the data.

Suppose that we have to go as far as 5th differences; differences after the 5th being negligible. Then, if θ is numerically small, *i.e.* if x_θ is near x_0, the values of the c's in terms of advancing differences are:

$$\left.\begin{aligned} c_0 &= u_0 \\ c_1 &= (\Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \tfrac{1}{4}\Delta^4 + \tfrac{1}{5}\Delta^5)\,u_0 \\ c_2 &= (\Delta^2 - \Delta^3 + \tfrac{11}{12}\Delta^4 - \tfrac{5}{6}\Delta^5)\,u_0 \\ c_3 &= (\Delta^3 - \tfrac{3}{2}\Delta^4 + \tfrac{7}{4}\Delta^5)\,u_0 \\ c_4 &= (\Delta^4 - 2\Delta^5)\,u_0 \\ c_5 &= \Delta^5 u_0 \end{aligned}\right\}.$$

In terms of central differences they are:

$$\left.\begin{aligned} c_0 &= u_0 \\ c_1 &= (\mu\delta - \tfrac{1}{6}\mu\delta^3 + \tfrac{1}{30}\mu\delta^5)\,u_0 \\ c_2 &= (\delta^2 - \tfrac{1}{12}\delta^4)\,u_0 \\ c_3 &= (\mu\delta^3 - \tfrac{1}{4}\mu\delta^5)\,u_0 \\ c_4 &= \delta^4 u_0 \\ c_5 &= \mu\delta^5 u_0 \end{aligned}\right\}.$$

* G. J. Lidstone, "Notes on Everett's Interpolation Formula", *Proc. Edinb. Math. Soc.* 40 (1921-22).

Of the two sets of formulae, I find the second are decidedly the most convenient to use. They also have the advantage that on the whole they give more accurate results.

22. *Sums (ordinary notation).* The beauty of the central-difference system cannot be fully realised without a brief study of central sums.

I will begin with the ordinary system of notation.

In any table of differences, arranged in columns in the ordinary way, the differences in any one column are the successive sums of the differences in the next column. Thus, in the example in § 13 (Table 3), we have, as regards the 2nd-difference column,

$$58+6=64,\quad 58+6+10=74,\quad 58+6+10+13=87,$$

and so on. Looking at the key arrangement in § 14, we see that

$$\Delta^2u_{15}+\Delta^3u_{15}=\Delta^2u_{16},\quad \Delta^2u_{15}+\Delta^3u_{15}+\Delta^3u_{16}=\Delta^2u_{17},\ \dots,$$

and, generally, $\Delta^2u_{15}+\Delta^3u_{15}+\Delta^3u_{16}+\dots+\Delta^3u_n=\Delta^2u_{n+1}$.

Thus the entries in the column marked Δ^2u can be built up by successive additions of the entries Δ^2u. In the ordinary notation, this process of addition is denoted by Σ; *i.e.* ignoring the initial terms (which are analogous to the arbitrary constant in integration), the sum

$$\dots\dots+\Delta^2u_{n-2}+\Delta^2u_{n-1}+\Delta^2u_n$$

is denoted by $\Sigma\Delta^2u$, with the proper suffix.

What is the suffix to be? We might naturally write the above sum as $\Sigma\Delta^2u_n$. But this would mean that $\Sigma\Delta^2u_n$ is equal to Δ^2u_{n+1}. Since the operations Δ and Σ cancel one another, the practice is that we denote

$$\dots\dots+v_{n-2}+v_{n-1}+v_n$$

by Σv_{n+1}, so that $\Sigma\Delta^2u_{n+1}$ means the sum of the series of 3rd differences up to and including the term next before Δ^2u_{n+1}. With this meaning, $\Sigma\Delta^2u_{n+1}$ (starting from a proper value) is always equal to Δ^2u_{n+1}, and the operations Δ and Σ are so related that $\Delta\Sigma=1$, $\Sigma\Delta=1$.

23. *Disadvantages of ordinary system.* This system, of denoting

$$\dots+v_{n-2}+v_{n-1}+v_n$$

by Σv_{n+1}, is consistent, but confusing. Consider, for example, the two series

$$\left.\begin{aligned} S_1&\equiv 1^2+2^2+3^2+\dots+(n-1)^2+n^2\\ S_2&\equiv (n+1)^2+(n+2)^2+\dots+100^2\end{aligned}\right\}.$$

We might naturally regard S_1 as a function of n, and represent it by some symbol which suggests this, *e.g.* by Σn^2. But the Δ system represents it by $\Sigma(n+1)^2-\Sigma 1^2$, which we should regard as a function of $n+1$.

Again, we might naturally regard S_2 as a function of $n+1$; but this is hardly consistent with our regarding S_1 as a function of n, since the two together make up $1^2+2^2+\dots+100^2$, which is independent of n.

The truth is that S_1 and S_2 ought both to be regarded as functions of $n+\frac{1}{2}$. We can easily verify this: for

$$S_1=\tfrac{1}{6}n(n+1)(2n+1)=\tfrac{1}{3}(n+\tfrac{1}{2})^3-\tfrac{1}{12}(n+\tfrac{1}{2}),$$

which contains only odd powers * of $n+\frac{1}{2}$.

Similarly

$$1^3+2^3+3^3+\dots+(n-1)^3+n^3=\{\tfrac{1}{2}n(n+1)\}^2=\{\tfrac{1}{2}(n+\tfrac{1}{2})^2-\tfrac{1}{8}\}^2,$$

which contains only even powers of $n+\frac{1}{2}$.

If we want an expression which shall be a neat function of n, we must halve the last term. We then have

$$\begin{aligned}1^2+2^2+\dots+(n-1)^2+\tfrac{1}{2}n^2&=S_1-\tfrac{1}{2}n^2\\&=\tfrac{1}{3}n^3+\tfrac{1}{6}n.\end{aligned}$$

* There would also, in the general case, be an absolute term, depending on the term with which the sum starts. In the present case this absolute term is 0.

Similarly $1^3+2^3+3^3+\ldots+(n-1)^3+\frac{1}{2}n^3=(\frac{1}{2}n^2+\frac{1}{2}n)^2-\frac{1}{2}n^3$
$=\frac{1}{4}n^4+\frac{1}{2}n^2.$

The first of these two expressions contains only odd powers of n, and the second contains only even powers.

24. *Central-sum notation.* The central-sum system of notation, constructed so as to be in harmony with the central-difference system, avoids the awkwardness that arises from the use of Σ. It introduces a summation-symbol σ which is exactly the inverse of δ, so that $\delta\sigma u_n=u_n$, etc. The result of this notation will be understood from Table 7.

Table 7.

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\ldots\sigma^2u_{17}=\Sigma^2u_{18}$	$\ldots\sigma u_{17\frac{1}{2}}=\Sigma u_{18}$	u_{17}	$\delta u_{17\frac{1}{2}}=\Delta u_{17}\ldots$	$\delta^2u_{17}=\Delta^2u_{16}\ldots$
$\ldots\sigma^2u_{18}=\Sigma^2u_{19}$	$\ldots\sigma u_{18\frac{1}{2}}=\Sigma u_{19}$	u_{18}	$\delta u_{18\frac{1}{2}}=\Delta u_{18}\ldots$	$\delta^2u_{18}=\Delta^2u_{17}\ldots$
$\ldots\sigma^2u_{19}=\Sigma^2u_{20}$	$\ldots\sigma u_{19\frac{1}{2}}=\Sigma u_{20}$	u_{19}	$\delta u_{19\frac{1}{2}}=\Delta u_{19}\ldots$	$\delta^2u_{19}=\Delta^2u_{18}\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Here the sum of the series $\ldots+u_{17}+u_{18}$ (which, we must remember, always contains an "arbitrary constant") is represented by $\sigma u_{18\frac{1}{2}}$, the sum of $\ldots+\sigma u_{17\frac{1}{2}}+\sigma u_{18\frac{1}{2}}$ by σ^2u_{19}, and so on.

The principle of the notation, and the effect of the operator μ, can be seen by the following example:

$$\ldots\ldots+(n-2)^2+(n-1)^2=\sigma(n-\tfrac{1}{2})^2,$$
$$\ldots\ldots+(n-2)^2+(n-1)^2+n^2=\sigma(n+\tfrac{1}{2})^2,$$
$$\ldots\ldots+(n-2)^2+(n-1)^2+\tfrac{1}{2}n^2=\text{mean of the above}$$
$$=\mu\sigma n^2.$$

Similarly $\frac{1}{2}n^2+(n+1)^2+\ldots+100^2=(\text{total sum up to } 100^2)-\mu\sigma n^2.$

Excursus II. Euler-Maclaurin and Allied Theorems.

25. We come now to a small group of theorems which are both important and interesting. The specially important one, for our purpose, is the Euler-Maclaurin Theorem: but I will take them in order of time.

26. *Gregory.* First, as in the case of interpolation, comes a formula by James Gregory (1670): I will consider this later (see § 34 (ii)).

27. *James Bernoulli.* Next comes the formula of James Bernoulli (published in 1713, after his death) for the sum of the powers of successive integers. I have already referred to the simple cases:

$$Sm\equiv 1+2+3+\ldots+(m-1)+m=\tfrac{1}{2}m(m+1);$$
$$Sm^2\equiv 1^2+2^2+3^2+\ldots+(m-1)^2+m^2=\tfrac{1}{3}m(m+1)(2m+1);$$
$$Sm^3\equiv 1^3+2^3+3^3+\ldots+(m-1)^3+m^3=\{\tfrac{1}{2}m(m+1)\}^2.$$

Bernoulli gives the general formula

$$Sm^p=\frac{1}{p+1}m^{p+1}+\tfrac{1}{2}m^p+\frac{p}{2}Am^{p-1}+\frac{p(p-1)(p-2)}{4!}Bm^{p-3}$$
$$+\frac{p\ldots(p-4)}{6!}Cm^{p-5}+\ldots,$$

the series ending with a term in m or m^2. In this formula A, $B\ldots$ are certain numerical coefficients, alternately positive and negative, which, all taken

positively, are now called Bernoulli's numbers; the first three of these numbers being $B_1 = +A = \frac{1}{6}$, $B_2 = -B = \frac{1}{30}$, $B_3 = +C = \frac{1}{42}$.

Bernoulli does not give a general proof of the theorem. He works out each case, up to $p = 10$, and arranges them consecutively, thus:

$$Sm = \tfrac{1}{2}m^2 + \tfrac{1}{2}m,$$
$$Sm^2 = \tfrac{1}{3}m^3 + \tfrac{1}{2}m^2 + \tfrac{1}{6}m,$$
$$Sm^3 = \tfrac{1}{4}m^4 + \tfrac{1}{2}m^3 + \tfrac{1}{4}m^2,$$
$$\vdots$$
$$Sm^{10} = \tfrac{1}{11}m^{11} + \tfrac{1}{2}m^{10} + \tfrac{5}{6}m^9 - m^7 + m^5 - \tfrac{1}{2}m^3 + \tfrac{5}{66}m.$$

He then examines the terms in each vertical column, and discovers the constancy, for successive values of p, of the coefficients denoted by A, B, C, D; and then states the general formula.

28. *Observations.* There are several points of interest in the formula.

(i) The first thing we notice is that the earlier part of it is rather crowded. We have terms in m^{p+1}, m^p, and m^{p-1}. After that the indices go by difference 2. The term in m^p seems out of place. But I have already dealt with that point in Excursus I, in connexion with Σ and σ. Here Sm^p means

$$\ldots + (m-1)^p + m^p.$$

I pointed out that this is a function of $m + \frac{1}{2}$, and that, if we want a neat expression in m, the series of terms must end up with $\frac{1}{2}m^p$, not with m^p. For a similar reason, since the sum is taken between limits, we ought to begin with $\frac{1}{2} \cdot 0^p$. Making these adjustments, by subtracting $\frac{1}{2}m^p$ from both sides, the formula becomes

$$\tfrac{1}{2} \cdot 0^p + 1^p + 2^p + \ldots + (m-1)^p + \tfrac{1}{2}m^p$$
$$= \frac{1}{p+1} \cdot m^{p+1} + B_1 \frac{p}{2!} m^{p-1} - B_2 \frac{p(p-1)(p-2)}{4!} m^{p-3} + \ldots .$$

This produces regularity in the indices.

(ii) The first term now claims our attention. It has $p+1$ in the denominator and does not seem to harmonise with the other terms. Also the factors

$$\frac{p}{2!}, \frac{p(p-1)(p-2)}{4!}, \ldots,$$

which occur in the 2nd, 3rd ... terms, just fail to be binomial coefficients. But let us put $1/(p+1)$ outside the whole thing, and see what happens. Writing $(p+1, t)$ for the binomial coefficient $(p+1)p \ldots (p-t+2)/t!$, and entering the numerical values of Bernoulli's numbers, we get

$$\tfrac{1}{2} \cdot 0^p + 1^p + 2^p + \ldots + (m-1)^p + \tfrac{1}{2}m^p$$
$$= \frac{1}{p+1} \{m^{p+1} + B_1 (p+1, 2) m^{p-1} - B_2 (p+1, 4) m^{p-3} + \ldots\}$$
$$= \frac{1}{p+1} \{m^{p+1} + \tfrac{1}{6} (p+1, 2) m^{p-1} - \tfrac{1}{30} (p+1, 4) m^{p-3} + \tfrac{1}{42} (p+1, 6) m^{p-5} - \ldots\};$$

the terms after the first being alternately positive and negative.

(iii) Finally, look at the last term. Is it not curious that this, for an odd value of p, should be the term ending in m^2, not the term ending in m^0?

This difficulty disappears when we have regard to the fact that we are taking the sum between limits: that we are taking what, by analogy with the nomenclature of integrals, we may call a *definite* sum, as distinct from the *indefinite* sum, which contains an arbitrary constant. This definite sum, in our notation, is

$$\mu\sigma m^p - \mu\sigma 0^p.$$

The two terms in this expression each contain the same arbitrary constant, which will include the term in m^0; and when we subtract the one from the other the term in m^0 will disappear.

29. *Euler and Maclaurin.* We now come to the general theorem, of which the preceding is a particular case. It was published independently by Euler in 1738, and by Colin Maclaurin, who was professor at Edinburgh, in 1742; but publication, in each case, was some years after discovery.

The data for use of the theorem are the $m+1$ values of a quantity u, corresponding to $m+1$ values of x proceeding by a constant interval h. Thus, if the x's are $x_0, x_1 \equiv x_0+h \ldots x_m \equiv x_0+mh$, the u's are $u_0, u_1 \ldots u_m$. The theorem states that

$$\int_{x_0}^{x_m} u\,dx = h\left(\tfrac{1}{2}u_0+u_1+u_2+\ldots+u_{m-1}+\tfrac{1}{2}u_m\right)+\Big[\phi(x)\Big]_{x=x_0}^{x=x_m}, \qquad (29.1)$$

where $\Big[\phi(x)\Big]_{x=x_0}^{x=x_m}$ denotes $\phi(x_m)-\phi(x_0)$, and $\phi(x)$ is a certain series of terms in h^2u^{I}, h^4u^{III}, $h^6u^{\mathrm{V}}\ldots$; u^{I}, u^{III}, $u^{\mathrm{V}}\ldots$ being the 1st, 3rd, 5th ... derivatives of u with regard to x. The integral on the left-hand side may be denoted by I; the first term on the right-hand side by C_1 (for reasons stated in § 31 (ii)); and the second term, which is in the nature of a remainder, by R_1. The formula thus becomes

$$I = C_1 + R_1.$$

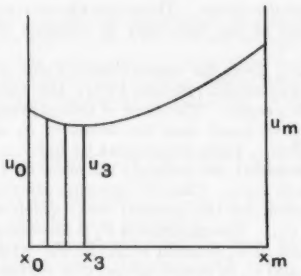


FIG. 3.

The meaning of this may be seen by reference to Fig. 3. There is a plane figure bounded by ordinates u_0 and u_m; and we know the magnitudes of a series of equidistant ordinates $u_0, u_1, u_2 \ldots u_m$, the common distance being h. Then I is the true area of the figure; C_1 is an approximate expression for the area, obtained by joining the tops of successive ordinates and adding the areas of the trapezia so constructed; and the theorem states that

$$I = C_1 + \phi(x_m) - \phi(x_0).$$

The expression for $\phi(x)$ is

$$\phi(x) = -\frac{B_1}{2!}h^2u^{\mathrm{I}} + \frac{B_2}{4!}h^4u^{\mathrm{III}} - \frac{B_3}{6!}h^6u^{\mathrm{V}} + \ldots$$

$$= -\tfrac{1}{12}h^2u^{\mathrm{I}} + \tfrac{1}{720}h^4u^{\mathrm{III}} - \tfrac{1}{30240}h^6u^{\mathrm{V}} + \ldots,$$

so that

$$R_1 = \Big[-\tfrac{1}{12}h^2u^{\mathrm{I}} + \tfrac{1}{720}h^4u^{\mathrm{III}} - \tfrac{1}{30240}h^6u^{\mathrm{V}} + \ldots\Big]_{x=x_0}^{x=x_m}.$$

30. *Aspects of the formula.* The theorem has various aspects.

(i) In the form in which I have stated it, it is a quadrature-formula. If u is an expression which we cannot integrate, the theorem gives a formula for calculating the integral of u from the sum of a series of values. If, for instance, we wished to calculate the logarithm of x (to base e), we could calculate a series

of values of $1/x$, add them, and multiply the sum by h; then, adding the corrective terms R_1, we should have the integral

$$\int_{x_0}^{x_m} \frac{1}{x}\,dx = \log x_m - \log x_0.$$

(ii) By transferring R_1 to the other side, we have

$$C_1 = I - R_1.$$

Thus we can find the sum of a series of terms by integration. This is, indeed, what Bernoulli's formula does. To find Sm^p we begin by integrating m^p with regard to m: this gives the term $m^{p+1}/(p+1)$ in Bernoulli's formula.

(iii) There is also another way in which we can look at the matter. We define the integral $\int_a^b u\,dx$ as the limit of a certain sum, obtained (if we use a graphical interpretation) by adding ordinates at equal intervals h and multiplying their sum by h, and then making the interval h become indefinitely small. In the formula, C_1 is the value obtained for a finite value of h, I is the limit as h approaches zero, and their difference $-R_1$ shows how near we have got to the limit.

31. *Nature of remainder-terms.* There are also certain things we can do by merely taking account of the fact that R_1 consists of terms in h^2u^{I}, h^4u^{III}, $h^6u^{\mathrm{V}}\ldots$.

(i) The expression C_1 gives the approximate value for I which is obtained by what is called the *trapezoidal rule* (see § 29); this value being usually called the *chordal area* (of the graph). The error of this expression is $-R_1$. Suppose that we have taken h so small that the terms of R_1 after the first (at each extremity) are negligible. Then we see that by halving h, *i.e.* by doubling the number of terms to be added, we multiply the error by 1/4, not merely by 1/2.

(ii) Suppose that m is even. Then by ignoring alternate terms u_1, u_3, ... we can get a new expression for the chordal area, which we can call C_2. (This explains the notation C_1.) The expression C_2 is less accurate, usually, than C_1; but the errors of C_1 and of C_2 contain terms in h^2u^{I} with different coefficients, and by combining C_1 and C_2 in proper proportion we can get rid of these terms and thus obtain a more accurate formula. Actually this formula is Simpson's rule, with corrections. A very large number of formulae can be obtained on this principle.

(iii) The principle can be extended to cases—not very frequent—in which the data are the mid-ordinates of a series of strips. In these cases the chordal area is replaced by the *tangential area*, obtained by adding the values of the mid-ordinates and multiplying the sum by the common breadth of the strips.

32. *Basis of the formula.* (i) The Euler-Maclaurin formula is the result of the addition of a number of component formulae for the successive strips. Thus for the first strip we have

$$\int_{x_0}^{x_1} u\,dx = \tfrac{1}{2}h\,(u_0+u_1) + \phi\,(x_1) - \phi\,(x_0). \qquad \ldots\ldots\ldots\ldots\ldots\ldots (32.1)$$

Similarly

$$\int_{x_1}^{x_2} u\,dx = \tfrac{1}{2}h\,(u_1+u_2) + \phi\,(x_2) - \phi\,(x_1),$$

$$\vdots$$

$$\int_{x_{m-1}}^{x_m} u\,dx = \tfrac{1}{2}h\,(u_{m-1}+u_m) + \phi\,(x_m) - \phi\,(x_{m-1}).$$

Adding all these together, we get (29.1). The important thing is that $\phi\,(x)$ is of the same form throughout, so that all the ϕ's cancel out except $-\phi\,(x_0)$ and $+\phi\,(x_m)$.

(ii) To see that this is possible, consider the first strip. Let $u \equiv f(x)$. Then

$$\int_{x_0}^{x_1} u\,dx = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(x_{\frac{1}{2}}+\xi)\,d\xi$$

$$= \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \left\{ f(x_{\frac{1}{2}}) + f^{\mathrm{I}}(x_{\frac{1}{2}})\,\xi + f^{\mathrm{II}}(x_{\frac{1}{2}})\frac{\xi^2}{2!} + \dots \right\} d\xi$$

$$= hf(x_{\frac{1}{2}}) + \text{terms in } h^3 f^{\mathrm{II}}(x_{\frac{1}{2}}),\ h^5 f^{\mathrm{IV}}(x_{\frac{1}{2}}), \text{ etc.} \quad \dots\dots\dots(32.2)$$

Also $\frac{1}{2}h(u_0+u_1) = \frac{1}{2}h\{f(x_{\frac{1}{2}}-\frac{1}{2}h)+f(x_{\frac{1}{2}}+\frac{1}{2}h)\}$

$$= hf(x_{\frac{1}{2}}) + \text{terms in } h^3 f^{\mathrm{II}}(x_{\frac{1}{2}}), \text{ etc.} \quad \dots\dots\dots(32.3)$$

The difference of the expressions in (32.2) and (32.3) consists of terms in $h^3 f^{\mathrm{II}}(x_{\frac{1}{2}})$ etc.; and it is easy to see that these can be expressed in the form $\phi(x_1) - \phi(x_0)$, where $\phi(x)$ consists of terms in $h^2 f^{\mathrm{I}}(x)$, $h^4 f^{\mathrm{III}}(x)$, ...

(iii) We must, however, have regard to the convergence or divergence of the series $\phi(x)$. If u is a polynomial, the series terminates, and the theorem is true exactly. If u is not a polynomial, we must not use the formula until the question of convergence has been considered; and we may find that it can only be used to a limited extent.

33. *Stirling's theorem.* A familiar example of the use of the Euler-Maclaurin theorem for obtaining an approximate result even though $\phi(x)$ may be divergent is Stirling's formula for the value of $n!$ or $\log n!$ when n is large. The formula, which was published by Stirling in 1730 (*i.e.* before the Euler-Maclaurin publication), is of importance in the theory of statistics. Using logarithms, it may be written

$$\log n! = \tfrac{1}{2}\log(2\pi) + (n+\tfrac{1}{2})\log n - n + \frac{1}{12n} + \psi,$$

when ψ is a small quantity whose value can be found within certain limits, these limits becoming smaller as n increases.

It will be noticed that we can simplify the form of the expression by removing $\frac{1}{2}\log n$ from both sides. We then have

$$\tfrac{1}{2}\log 1 + \log 2 + \dots + \log(n-1) + \tfrac{1}{2}\log n = \tfrac{1}{2}\log(2\pi) + n\log n - n + \frac{1}{12n} + \psi.$$

34. *Euler-Maclaurin formula in terms of ordinates: Gregory's formula.*

(i) I have so far been considering the Euler-Maclaurin formula as a purely algebraical expression: its applications being based on the assumption that the derivatives of u, for the extreme values of x, are known or can be found. But in actual practice we do not usually know the derivatives directly: we can only deduce them, more or less exactly, from the given values of u. Ordinarily it is convenient to express the derivatives in terms of differences of the u's; and, since R_1 is a definite function of the derivatives, we can save a step by expressing R_1 generally in terms of the differences.

(ii) I have already mentioned a formula by James Gregory, as the precursor of the Euler-Maclaurin formula. Gregory's is purely a quadrature formula, and may be written:

$$\text{Area} = h(\tfrac{1}{2}u_0 + u_1 + u_2 + \dots + u_{m-1} + \tfrac{1}{2}u_m) + R_1',$$

where R_1' involves two series of differences, the one derived from u_0, u_1, u_2 ... and the other from u_m, u_{m-1}, u_{m-2} ... . The complete expression will be found in the books.

It may be asked why Gregory did not express his formula in terms of derivatives, the expression in terms of differences being regarded as a secondary one. The answer is that this was in 1670, just before the period which saw the

development of the infinitesimal calculus; and Gregory died five years later. The dates of birth are interesting:

1638 Gregory,
1642 Newton,
1646 Leibnitz.

(iii) For actual calculation, Gregory's formula is not very convenient. In the Euler-Maclaurin formula, R_1 involves derivatives of odd order only: and we know that a derivative of odd order can be expressed in terms of central differences of odd order. This is therefore eminently a case for central differences. Gregory's formula for R_1, if we go to 4th differences, is

$$-\tfrac{1}{12}(\Delta u_{m-1}-\Delta u_0)-\tfrac{1}{24}(\Delta^2 u_{m-2}+\Delta^2 u_0)-\tfrac{19}{720}(\Delta^3 u_{m-3}-\Delta^3 u_0)$$
$$-\tfrac{3}{160}(\Delta^4 u_{m-4}+\Delta^4 u_0).$$

This has eight terms. The central-difference formula for $\phi(x)$ is

$$\phi(x)=h(-\tfrac{1}{12}\mu\delta u+\tfrac{11}{720}\mu\delta^3 u-\ldots),$$

so that the central-difference formula for R_1 only contains four terms.

(iv) On the other hand, there are some cases in which it is better to adhere to derivatives. The leading term in $\phi(x)$ is $-\tfrac{1}{12}h^2u^{\mathrm{I}}$; and it may be possible to get a better value for hu^{I}, at one or both extremities, by introducing an auxiliary function than by using the ordinary expression in terms of central differences. This is a branch of the subject with which I cannot deal here.

35. *Digression: use of central-difference formula at ends of range.* At this point it seems desirable to refer to a difficulty which may have occurred to some of you. If we only know the values within a limited range, how can we use a central-difference formula, which involves values lying outside this range?

The answer is that there is a distinction—which writers of text-books sometimes overlook—between "using central differences" and "using a central-difference formula". I will illustrate the distinction by taking a particular case.

Table 8.

x	u				
		(624)		(4)	
86° 10′	14·924		(57)		(3)
		681		(7)	
20′	15·605		64		(3)
		745		10	
30′	16·350		74		3
		819		13	
40′	17·169		87		
		906			
50′	18·075				

Table 8 (disregarding for the moment the figures in brackets) is identical with part of Table 3 in § 13. Here x is an angle, h is 10′, and the table gives the values of $u \equiv \tan x$ for 5 successive values of x, and the differences of these u's up to the 4th difference, which is as far as we need to go. Suppose that the first value of u that is given to us is for $x=86°\ 10'$, and that we want to find hu^{I} for this value of x.

The appropriate formula, according to the ordinary method, gives (omitting the factor ·001)

$$\begin{aligned} hu^{\mathrm{I}} &= (\Delta-\tfrac{1}{2}\Delta^2+\tfrac{1}{3}\Delta^3-\tfrac{1}{4}\Delta^4)u \\ &= 681-32+\tfrac{10}{3}-\tfrac{3}{4} \\ &= 651\tfrac{7}{12}. \end{aligned}$$

This is the best result we can get from the data. But now let us take the 4th-difference 3 to be constant, and continue the table backwards on this basis.

This gives us the additional differences, entered in brackets; and we can apply central-difference formulae to this table, though the differences in it are not the true central differences. Constructing means in the usual way, we find that the central differences to be used are

$$652\tfrac{1}{2} \quad 57 \quad 5\tfrac{1}{2} \quad 3;$$

and the deduced value of hu^1 is

$$\begin{aligned} hu^1{}_0 &= \mu\delta u_0 - \tfrac{1}{6}\mu\delta^2 u_0 \\ &= 652\tfrac{1}{2} - 1\tfrac{1}{2} \\ &= 651\tfrac{7}{12}, \end{aligned}$$

which is the same as we obtained above.

As a general rule, it is better to have one simple set of formulae applicable to all cases than to have different formulae for different cases.

36. *Mutual relation of Bernoulli's and Euler-Maclaurin formulae.* I have said (§ 29) that Bernoulli's sum-formula is a special case of the Euler-Maclaurin formula. It is easy to verify this by writing $u \equiv m^p$. But, if we limit ourselves to simple polynomials, we could regard Bernoulli's as the primary formula, the Euler-Maclaurin being deduced from it by a re-grouping of terms.

W. F. SHEPPARD.

GLEANINGS FAR AND NEAR.

789. . . . At last one of them caught sight of a vast seated assemblage of elliptical rhomboids which was wooing the public under the name of Venus. "Oh! my dear," she said, "here she is! Here's the Venus!" And putting her head on one side she added: "Isn't she a pet!"—John Galsworthy (Authors' Society Dinner, Nov. 20, 1929.)

790. Macfarlane Gray . . . was really the inventor of steam steering-gear for ships. He was a great mathematician, and his power of concentration comparable with that of Socrates, for he is said to have held up the traffic at the Bank crossing while absorbed in thinking out some problem in mathematics.—R. E. Crompton, *Reminiscences*, 1928, p. 78.

791. "I can't think," said [Bernardin de St. Pierre] to Bonaparte, "why the savants have taken no notice of my 'Theory of the Tides'." "Do you know anything of the Differential Calculus?" asked the Consul.—"Nothing."—"Well, then, go and study it, and you will be able to answer yourself."—*Athenaeum*, 1857, p. 1270.

792. . . . One Isaac Newton, an instrument maker, formerly living near Leicester-fields, and afterwards a workman in the mint at the Tower, might possibly pretend to vie with me for fame in future times. The man, it seems, was knighted for making sun-dials better than others of his trade, and was thought to be a conjurer, because he knew how to draw lines and circles upon a slate, which nobody could understand. But adieu to all noble attempts for endless renown, if the ghost of an obscure mechanic shall be raised up to enter into competition with me, only for his skill in making pot-hooks and hangers with a pencil, which many thousand accomplished gentlemen and ladies can perform as well with pen and ink upon a piece of paper, and in a manner as little intelligible as those of Sir Isaac.—[Dean Swift's only mention of Newton.] *A Complete Collection of Genteel and Ingenious Conversation*, by Simon Wagstaff. *The Works of Swift* (Bell & Sons, 1880), vol. 2, p. 332.

793. We are strongly inclined to suppose that [Horrocks] must have been twenty-three years old in 1639, instead of twenty, when he began to do duty at Hoole. It makes no difference as to his fame; but we must remember, that though genius may be of any age, learning must have time.—*Loc. cit.* 781.

THE METHODS OF CONSTRUCTION OF THE EARLIEST TABLES OF LOGARITHMS.*

By J. Henderson, Ph.D.

The first table of logarithms was published by John Napier in 1614 under the title of *Mirifici Logarithmorum Canonis Descriptio ...*, at Edinburgh, but the explanation of the construction was not then given. The author, however, promised to publish this later, when he had been assured that the leading mathematicians of the day approved of his work. Any doubt on this point was very soon removed for, within a short time, Edward Wright, who, after studying at Cambridge, had devoted himself to navigation, was engaged in translating the *Descriptio* into English. Henry Briggs, Gresham Professor in London since 1596, lectured to his students on the work of Napier as soon as he had grasped its significance, and he took it up with great enthusiasm. Soon he was in communication with the inventor and even travelled twice to Scotland in the summers of 1615 and 1616, before Napier's death, which took place in 1617. The *Descriptio* seems to have taken some time to reach the Continent, for it was not till 1617 that Kepler first saw a copy. He apparently did not realise its importance at that time, but in the following year he became so enthusiastic that he wrote a letter to Napier (*i.e.* after the latter's death) expressing his admiration and hoping for the publication of the method of construction of the Canon. In 1619 the posthumous work of Napier appeared with the title, *Mirifici Logarithmorum Canonis Constructio*. The construction of the table of logarithms is fully given and the exposition of the procedure, step by step, is exceedingly simple and beautiful. This work was admirably translated by W. R. Macdonald in 1889†.

Throughout this work Napier is mainly concerned with the simplification of trigonometrical calculations, and his table contains the logarithms of sines, with the radius taken to be 10^7. After defining arithmetical and geometrical progressions he proceeds to calculate various geometrical progressions, which are to be used later in calculating the logarithms of sines. Three tables are formed :

First Table. This contains 101 proportionals, of which the first is 10000000·0000000, the ratio being 10000000 : 9999999, and the last proportional is 9999900·0004950. These proportionals are easily formed each in succession from the preceding one by subtracting $1/10^7$ of its value. Simple subtractions only are necessary.

Second Table. This table contains 51 proportionals, whose first member is 10000000·000000 and whose ratio in integers is as nearly as possible the ratio of the first and last terms of Table I. Napier takes this ratio to be 100,000 : 99999 in place of $10^7 : 9999900·000495$. This set of proportionals is also formed by simple subtractions, since each member is formed by subtracting $1/10^5$ of the preceding. The last term is 9995001·222927. [An error occurred in this table and this decimal should be ·224804.]

Third Table. This consists of 69 columns and in each column there are 21 proportionals, whose ratio is nearly equal to the ratio of the first term to the last in the second table. Napier takes this ratio to be 10000 to 9995 instead of $10^7 : 9995001·222927$. The calculations here are also simple, as it is merely necessary to subtract continuously the 2000th part of the preceding term. The first term of the first column is 10000000·00000 and the last is 9900473·57808. The first numbers of all the columns proceed in the ratio

* A paper read at the Bristol meeting of the British Association, September, 1930.

† *The Construction of the Wonderful Canon of Logarithms by John Napier, Baron of Merchiston*. Translated from Latin into English with Notes. Edinburgh and London, 1889. (Blackwood and Sons.)

of 100 : 99, which is nearly equal to the ratio of the first term to the last in the first column. These numbers are again easily formed. Similarly the second, third, ... terms in the following columns form progressions whose first terms are the second, third, ... terms respectively in the first column and whose ratio is 100 : 99. The table thus consists of 69 columns of 21 rows, such that the ratio of the geometrical progression in the columns is 10000 : 9995, while the ratio in the rows is 100 : 99. In the 69th column the first term is 5048858·8900 and the last is 4998609·4034. Napier has thus obtained a large number of proportionals between the radius (10^7) and half radius. He then tackles the problem of supplying logarithms corresponding to each of those proportionals and proceeds from a few definitions.

1. To increase arithmetically is in equal times to be augmented by a quantity always the same. [This requires no amplification.]

2. To decrease geometrically is this, that in equal times, first the whole quantity, then each of its successive remainders is diminished, always by a like proportional part.

3. Whence a geometrical moving point approaching a fixed one has its velocities proportionate to its distances from the fixed one.

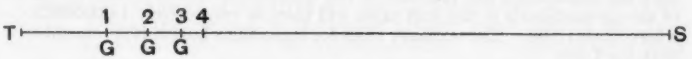


FIG. 1.

Let TS be the radius (using Napier's expression). Let the point G travel in the direction S, so that in equal times it is borne from T to 1, which for example may be the tenth part of TS, and from 1 to 2, the tenth part of $1S$, and from 2 to 3, the tenth part of $2S$, ... and so on. Then TS, $1S$, $2S$, $3S$ ... are said to decrease geometrically. The velocity of the moving point will be directly as the distances traversed in equal times. The ratio of the distances traversed in equal times, $T1$, 12, 23, ... is that of the distances TS, $1S$, $2S$, ... Hence it follows that the ratio of the distances of G from S, viz. TS, $1S$, $2S$, $3S$, ... is the same as that of the velocities of G at the points T, 1, 2, 3, ... Definition of logarithm is then given. *The logarithm of a given sine is that number which has increased arithmetically with the same velocity throughout as that with which the radius began to decrease geometrically and in the same time as radius has decreased to the given sine.*

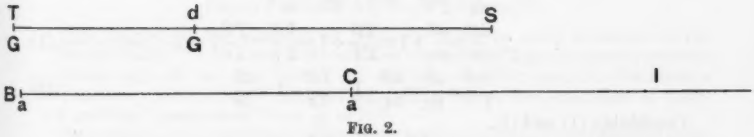


FIG. 2.

Let TS be the radius and dS a given sine; let G move geometrically from T to d in certain determinate moments of time. Let BI be another line, "infinite towards I", along which let a move arithmetically from B with the same velocity as G had at T. Let a reach C in the same time that G takes to reach d. Then BC is called the logarithm of the sine dS.

It is then obvious that (i) $\log(\text{radius}) = 0$;

(ii) $\log(\text{any sine}) > (\text{radius} - \text{sine})$

since $BC > Td$.

Consider a point O on TS to the left of T, such that $OS : TS = TS : dS$; then

$OT > BC > Td$, since velocity for OT is greater than the velocity of a and the times are the same.

$$\therefore \log(dS) < OT, \text{ i.e. } < OS - TS, \text{ i.e. } < \frac{TS}{dS} \cdot TS - TS,$$

$$\text{i.e. } < \frac{TS}{dS}(TS - dS).$$

If we write r for the radius and s for sine, we have

$$(r-s) < \log s < \frac{r}{s}(r-s).$$

Napier then proceeds to find the logarithm of his first proportional 9999999; its logarithm must lie between

$$1{\cdot}0000000 \quad \text{and} \quad \frac{10^7}{9999999} \times 1{\cdot}0000000,$$

i.e. between 1·0000000 and 1·0000001, and so we may take

$$\log(9999999) = 1{\cdot}00000005,$$

the mean of the two limits. The system is now defined and it will be noticed that the logarithms will increase as the numbers decrease. The logarithms of the proportionals in the first table will then be respectively $1{\cdot}00000005 \times n$ where $n = 0(1)100$. The ordinary rules for logarithms hold if it is remembered that $\log 1 \neq 0$.

The next important proposition deals with the difference of the logarithms of two sines. *The difference of the logarithms of two sines lies between two limits; the greater limit being to the radius as the difference of the sines to the smaller sine and the lower limit being to the radius as the difference of the sines to the greater sine.*

In symbols this means that if σ and s are two sines with $\sigma > s$, and r is the radius, then

$$\frac{\sigma - s}{s} r > \log \sigma \sim \log s > \frac{\sigma - s}{\sigma} r.$$

FIG. 3.

Proof: Let TS be the radius, dS the greater of the sines and eS the smaller sine. Beyond T take V, such that $ST : TV$ is $eS : de$. Let c be such that $TS : Tc$ is $dS : de$. Then

$$\frac{eS}{de} = \frac{TS}{TV} \quad \text{and} \quad \frac{TS}{Tc} = \frac{dS}{de};$$

$$\therefore \frac{dS}{de} = \frac{eS}{de} + 1 = \frac{TS}{TV} + 1 = \frac{VS}{TV} = \frac{TS}{Tc}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{TS}{TV} = \frac{eS}{de} = \frac{dS}{de} - 1 = \frac{TS}{Tc} - 1 = \frac{cS}{Tc}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Combining (1) and (2),

$$\frac{VS}{TS} = \frac{TV}{Tc} = \frac{TS}{cS}.$$

Now $eS : de = cS : Tc$ and $dS : de = TS : Tc$;

$$\therefore TS : cS = dS : eS.$$

$$\therefore \log(dS) - \log(eS) = \log(TS) - \log(cS). \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Since $\log(\text{radius}) = 0$, the right-hand side of (3) is numerically $\log(cS)$. By an earlier proposition

$$VT > \log(cS) > Tc;$$

$$\therefore VT > \log(dS) \sim \log(eS) > Tc.$$

This is the result required.

By means of this relation it may be seen that the logarithm of a sine can be obtained within close limits from the logarithm of another sine if the sines are not too far apart. The above inequality enables us to find the logarithm of 9999900·, the first proportional in the second table from the last one in the first table 9999900·000495, whose logarithm is already known. It is found that the logarithm of 9999900· lies between 100·0005050 and 100·0004950. The limits between which lie the logarithms of all the proportionals in the second table are then easily found, the logarithm of the last lying between 5000·0252500 and 5000·0247500. For the actual values of the logarithms we may take the mean of the limits.

The logarithms of the proportionals in the third table can be found by the use of the same inequality with sufficient accuracy as the calculations are carried out to 7 more figures than are to be retained ultimately in the table. The third table with the logarithms of the proportionals added is called by Napier "The Radical Table". The logarithms are given to one figure after the decimal point; if the second figure after the point is > 4, then the figure after the point is raised by a unit. The Radical Table is now sufficient for the complete calculation of the logarithmic table of sines.

To find the logarithms of all sines within the limits of the Radical Table: Multiply the difference of the given sine and the nearest tabular value by the radius. Divide the product by the easiest divisor, which may be the given sine, or the tabular value, or a value between those. The upper or lower limit of the difference of the logarithms or some intermediate value is obtained and none of those will differ sensibly from the true difference of the logarithms. Add this value to or subtract it from the logarithm of the tabular value according as the given sine is less than, or greater than, the tabular value.

Napier then calculated the logarithms of the ratios 2:1, 10:1 and combinations of those. To compute the logarithms of the sines not within the range of the Radical Table, he multiplied by a simple factor, such as 2, 4, 10, 100, etc., to bring the sines within this range. The logarithm of this new value was then found as above, and the required logarithm was then obtained by *adding* the logarithm of the factor used. In this way Napier completed his epoch-making table. We can but marvel at the simplicity and ingenuity of it and admire the man, who had the will to carry out the large number of calculations necessary without the aid of tables or computing machines.

The table, as finally printed, gives the logarithms of the sines and cosines to 7 figures (for the greater part of the table) for every minute of the quadrant. For verification purposes Napier used the theorem, expressed in modern terms,

$$\log \sin \theta - \log 2 = \log \sin \tfrac{1}{2}\theta + \log \cos \tfrac{1}{2}\theta.$$

With regard to the accuracy of Napier's table, an error occurred in the "Second Table". As a result the error in the table of logarithms may amount to three units in the last figure. In an Appendix the inventor describes a method for constructing a table of logarithms where the logarithm of unity is 0 and the logarithm of 10 or $\frac{1}{10}$ is 10^{10}.

Historically, it is important to note that in the *Constructio* the decimal notation is used with ease and power practically for the first time.

In 1624 Henry Briggs published the table which was the result of several years' strenuous work under the title *Arithmetica Logarithmica*. The logarithms of this table are not of the same type as Napier's. Briggs had suggested to Napier that it would be more convenient to take the logarithm of the whole sine to be 0, as in Napier's table, and the logarithm of a tenth of the whole sine (*i.e.* the sine of 5° 44′ 21″) to be 10^{10}. When Briggs visited Napier, the latter said that he had already observed the advantages of this change, but he considered that it would be still better to have the logarithm of unity equal to 0 and the logarithm of the whole sine equal to 10^{10}. Briggs at once saw the great advantage of this change and decided to put aside the calculations already

made, in order to devote himself to the calculation of logarithms according to the new system. This system is the most important practically and may be regarded as having the base 10. Briggs's table gives the logarithms to 14 decimal places with the characteristics of all numbers (not "sines") from 1 to 20,000 and from 90,000 to 100,000. He employed a combination of several methods in his calculations.

In the first place he noticed that the logarithm of a number of n digits lies between the limits $n-1$ and n. On this principle he devised a means of calculating the logarithms of the lower prime numbers, such as 2, 3, 5, ..., etc. He raised the number to ever increasing powers, and by noting the number of digits in the power and dividing it by the index he was able to find closer and closer limits for the logarithm of the prime.

To find $\log 2$:

$2^{10} = 1{,}024$, $3 < \log 1024 < 4$; $\therefore$ $\log 2$ lies between ·3 and ·4.

$2^{20} = 1{,}048{,}576$, $6 < \log 1{,}048{,}576 < 7$; $\therefore$ $\log 2$ lies between ·3 and ·35.

2^{100} is a number of 31 digits; $\therefore$ $\log 2$ lies between ·30 and ·31.

It is not necessary to calculate the powers fully; the computer need only retain a number of the leading digits and use the fact that the product of two numbers of m and n digits respectively has $(m+n)$ digits, except in those cases where the product of the leading figures plus the "carried" figure < 10, the product in that case having $m+n-1$ digits.

The following table will indicate how straightforward and simple the process is, although it is obvious that without the use of a calculating machine it is rather laborious. Five digits only are retained in the higher powers.

Leading digits of powers of 2.	Indices.	Number of digits.
2	1	1
4	2	1
16	4	2
256	8	3
1024	10*	4
10486	20	7
10995	40	13
12089	80	25
12676	100*	31
16069	200	61
25823	400	121
66680	800	241
10715	1000*	302

The powers marked by an asterisk, *i.e.* those whose indices are powers of 10, are formed by squaring the former such power three times in succession and then multiplying the third of those by the first. The work is carried out by a contracted multiplication process. The number of digits in each product is easily estimated. Ultimately Briggs found the logarithms of 2 and the smaller primes to 14 decimal places.

For the calculation of the logarithms of larger primes the following method was used. The square root of the given number was found and then the square root of the result; the process of root extraction was continued until the root was unity with as many ciphers after the decimal point as there are figures required in the logarithm and as many more decimal places were retained. Briggs discovered that the decimal parts of numbers such as this are directly proportional to the logarithms of these numbers. He extracted the square root of 10 continuously and stopped when he found that the decimal part of the last root was exactly half of that of the preceding to 32 decimal places. Since the logarithm of 10 was taken to be unity, he divided unity by 2 continuously

and found the corresponding logarithms. At this stage the decimal part is proportional to the logarithm and so it was now an easy matter to find the log of $1\cdot(000)^{51}1$ which he found to be

$$\cdot(0)^{16}43429448\,19,03251,804.$$

In this process 54 extractions of the square root were necessary; a very laborious proceeding. From the logarithm of $1\cdot(0)^{15}1$ the logarithm of the last square root of the given number was found by direct proportion, and the logarithm of the original prime number was obtained by multiplying this result by 2 raised to the power, whose index is the number of extractions which were necessary. The number of square root extractions necessary may be lessened considerably by raising the prime to a power whose first digit is 1 followed by one or more ciphers. This power, with the decimal point placed after the first digit *, is then operated on. The process is as before, with slight modifications in obtaining the final logarithm.

In order to lessen the labour involved in the extractions of the square roots, Briggs devised an ingenious method depending on differences. This is the earliest work on the "Differential Method" which I have seen, and it would appear that Briggs is the inventor of this powerful method. He observed that in the square root of a number $1+x$ where $0<x<1$, the decimal part becomes more and more nearly equal to $\frac{1}{2}x$ as x decreases. It is to be noted that Briggs did not know the series for $(1+x)^{1/2}$. In general he noticed that the decimal part was exactly half to as many significant figures as there were ciphers after the decimal point. When he subtracted the decimal part A of one root from half that of the preceding he noticed that this difference B was nearly equal to a quarter of the preceding B. [He calls B the "second difference".] It was then seen that the "third difference" C, between a B and $\frac{1}{4}$ of the preceding B, was nearly equal to $\frac{1}{8}$ of the preceding C, and so on. Briggs extracted the square roots continuously until the "fifth difference" E was exactly $\frac{1}{32}$ part of the preceding E. The succeeding square roots were then obtained by simple processes of addition, subtraction and division. The new values of E, D, C, B, A were easily calculated from the preceding values and as the process was continued these differences successively became non-effective until only the first was left. This method lessens the labour and I think would be much more conducive to accuracy. In his introduction Briggs illustrates this process by giving an example: to determine the logarithm of 6 he raises it to the ninth power and obtains 10,077,696. The operation of square root extraction is then commenced on the number 1·0077696 and it is only necessary to extract the square root 9 times to obtain a value of E which is $\frac{1}{512}$ of the preceding E.

By calculating the logarithms of a large number of primes and adding suitable logarithms to obtain the logarithms of composite numbers he formed a skeleton table of consecutive numbers with their logarithms. He then interpolated for numbers intermediate between two of the entries of the skeleton table and there is no doubt that he completely understood the method of differences. Unfortunately, he does not say how he was led to this method, and he does not explain its principles. Two cases are considered, (1) where the second differences are equal or nearly equal, (2) where there are any number of differences. The problem is: Given two numbers in the skeleton table with their logarithms, to find the logarithms of 9 arithmetical means between the given numbers. The method is essentially the same as that expressed in Newton's Interpolation Formula. In case (1), let a and b be the first and second differences in the skeleton table; Briggs formed the successive first differences for the diminished interval by adding to $\frac{1}{10}a$ the values

$$\begin{array}{lllll} \frac{45}{1000}b, & \frac{35}{1000}b, & \frac{25}{1000}b, & \frac{15}{1000}b, & \frac{5}{1000}b, \\ -\frac{5}{1000}b, & -\frac{15}{1000}b, & -\frac{25}{1000}b, & -\frac{35}{1000}b, & -\frac{45}{1000}b. \end{array}$$

* The number is divided by an appropriate power of 10.

These differences were then added to the preceding logarithms as they were formed.

In modern terms this process is equivalent to interpolation by means of the formula

$$f(x+n\omega)=f(x)+na+\frac{n(n-1)}{1.2}b+\frac{n(n-1)(n-2)}{1.2.3}c+\dots,$$

where $a, b, c, \dots$ are the original first, second, third ... differences. If the second differences are equal or nearly equal, this formula becomes

$$f(x+n\omega)=f(x)+na+\frac{n(n-1)}{1.2}b,$$

where ω is the interval of the argument. To insert the logarithms of 9 arithmetical means the values of n are taken to be ·1, ·2, ... ·9 in succession.

Let $f(x)=A$. The terms of the series of logarithms, deduced from the formula, will then be as follows:

A.	First Differences.
$A+\frac{1}{10}a-\frac{9}{200}b$	$\frac{1}{10}a-\frac{9}{200}b$
$A+\frac{2}{10}a-\frac{16}{200}b$	$\frac{1}{10}a-\frac{7}{200}b$
$A+\frac{3}{10}a-\frac{21}{200}b$	$\frac{1}{10}a-\frac{5}{200}b$
$A+\frac{4}{10}a-\frac{24}{200}b$	$\frac{1}{10}a-\frac{3}{200}b$
$A+\frac{5}{10}a-\frac{25}{200}b$	$\frac{1}{10}a-\frac{1}{200}b$
$A+\frac{6}{10}a-\frac{24}{200}b$	$\frac{1}{10}a+\frac{1}{200}b$
$A+\frac{7}{10}a-\frac{21}{200}b$	$\frac{1}{10}a+\frac{3}{200}b$
$A+\frac{8}{10}a-\frac{16}{200}b$	$\frac{1}{10}a+\frac{5}{200}b$
$A+\frac{9}{10}a-\frac{9}{200}b$	$\frac{1}{10}a+\frac{7}{200}b$
$A+a$	$\frac{1}{10}a+\frac{9}{200}b$

If the coefficients of b in the difference column are converted into fractions with 1000 as denominator, this table will correspond exactly to that used in Briggs's method, since the second differences in his skeleton table are negative.

If the second differences of his skeleton table were not too widely separated, he operated with the mean of the two differences at the beginning and end of the range within which he wished to interpolate. In an example given the third difference is 99, and by taking the mean of the second differences the maximum error will be ·8 unit of the last decimal place to which he calculated; this maximum will occur at $n=\cdot2$ and $n=\cdot8$. The logarithm of the middle arithmetic mean will be given exactly. If the third differences are less than 63, the maximum error involved in this method of interpolation is less than half a unit in the last place.

In his introduction Briggs goes on to describe a method for the calculation of the logarithms of the numbers between 20,000 and 90,000. This consists in finding the logarithms of numbers at intervals of 5 from 20,000 onwards by adding log 5 to the logarithms of 4001, 4002, 4003, ... It is then necessary to interpolate the logarithms of 4 arithmetic means between 20,000 and 20,005, between 20,005 and 20,010, and so on, by means of differences. In principle, the method is similar to that described above for the insertion of 9 arithmetical means.

The methods of Napier and Briggs had nothing in common; this was quite natural, for, when they commenced work on their tables, they did not regard logarithms in the same way. It has been pointed out above that the base of Briggs's system is 10; Napier's system is practically equivalent to a system whose base is $e^{-1}(e=2\cdot71828\dots)$. More accurately, if N is a number and L its Napierian logarithm, $N=10^7e^{-L/10^7}$.

J. HENDERSON.

MATHEMATICAL NOTES.

972. [K¹. 11. e.] *Quidde's Theorem on Concurrent Triads of Common Tangents.*

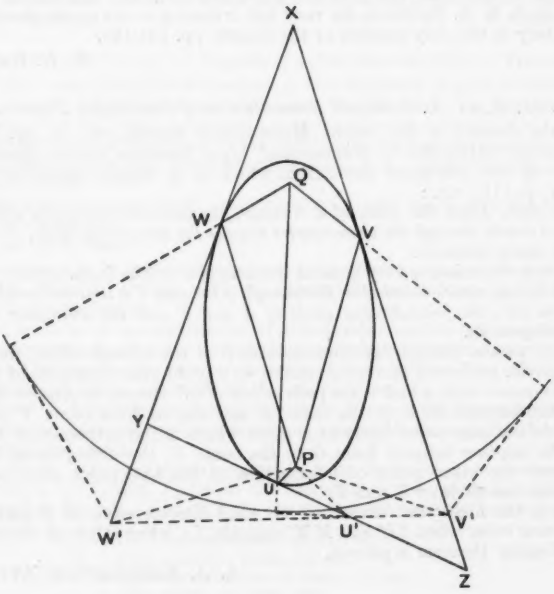
I am indebted to two correspondents for further references on this subject.

Mr. H. G. Forder points out that Coolidge's *Treatise on the Circle and the Sphere* does contain an exact version of Quidde's theorem, in the chapter on the oriented circle, where it appears (p. 364 ; Th. 11) as one of a group of theorems extracted from a paper by E. Müller (*Jahresbericht der deutschen Math.-Verein.*, vol. 20, p. 168 ; 1911). We have here an excellent example of the manner in which the orientation of lines and circles sometimes effects a kind of selective precision.

From Mr. C. E. Youngman I learn that, far from Quidde's own proof being unknown, the question in the *Nouvelles Annales* is taken from a paper by Quidde in Grunert's *Archiv* (vol. 15, p. 197 ; 1850), in which this is the first theorem. Quidde's proof is substantially the same as Mannheim's and Hart's, and he anticipates Hart not only in the result but in the use of the result as a step in a geometrical proof of Steiner's construction for the solution of Malfatti's problem. E. H. NEVILLE.

973. [K¹. 12. a ; L¹. 16. a.] *To construct three circles, having three given points X, Y, Z as centres, such that one common tangent of each pair of circles may pass through a fourth given point P.*

Draw an arbitrary line PW through P and suppose it to be the common tangent of the circles with centres X and Y ; the circle with centre X is



defined by the fact that it must touch PW, and the second tangent PV from P to this circle, which must be a common tangent of the circles with centres X and Z, is also defined; PV and PW are lines through P equally inclined to PX. Further, since PV touches the circle with centre Z and PW that with centre Y, these circles are defined as well as the second tangents PU_1 and PU_2 from P to these circles. To solve the problem we require positions of PW for which PU_1 and PU_2 coincide.

But it is clear that, if the direction of PW undergoes a change, PU_1 and PU_2 turn through equal angles in opposite senses: therefore there are two directions perpendicular to one another in which they coincide, and no more. The problem has two solutions, and if the lines PU, PV, PW give one solution the perpendicular lines through P give the other.

To find the two solutions. Let a conic with focus P touching YZ, ZX, XY be drawn, and let U, V, W be the points of contact. Since the two tangents XV and XW subtend equal angles at the focus P, a circle with centre X can be drawn to touch PV and PW; similarly for Y and Z. A solution has therefore been obtained. U, V, W are centres of similitude of the circles which are not collinear, and the second common tangents of the circles which pass through U, V, W meet in Q the second focus of the conic.

For the second solution draw lines PU', PV', PW' at right angles to PU, PV, PW; a circle with centre X can be drawn to touch PV' and PW', and similarly for Y and Z. But here U', V', W' are collinear, for they lie upon the directrix of the conic; they are three collinear centres of similitude of the circles; and the second common tangents of the circles which pass through U', V', W' cannot be concurrent, for the perpendicular distance of Q from any one of them is equal to the major axis of the conic.

It will be seen that this demonstration bears out the conclusions arrived at by Prof. E. H. Neville in his very full critical account of the theorem and its history in the July number of the *Gazette*, pp. 134-137.

H. W. RICHMOND.

974. [L¹. 19. a.] *L. J. Rogers' Generalisation of Feuerbach's Theorem.*

In the *Journal of the Indian Mathematical Society*, vol. iv, pp. 213-216 (December, 1912), Mr. V. Ramaswami Aiyar discusses certain general properties of the orthopolar circle from which L. J. Rogers' theorem (*Gazette*, vol. xv, p. 111), viz.:

If a conic touch the sides of a triangle, its joint-director circle with either confocal drawn through the circumcentre touches the nine-point circle,

can be easily deduced.

We first enunciate a very general theorem due to Mr. V. R. Aiyar:

If X be any conic inscribed in the triangle ABC and Y a confocal to it touching any line LL', the joint-director circle of X and Y cuts the orthopolar circle of LL' orthogonally.

If LL' passes through the circumcentre O of the triangle ABC, the orthopolar circle, as is well known, dwindles to a point (the orthopole of LL') on the nine-point circle which is the pedal circle of O. Hence, by Aiyar's theorem, the joint-director circle of the conic X and the confocal conic Y touching LL' cuts the nine-point circle at a point which is the orthopole of LL'. If MM' be another tangent from O to the conic Y, the orthopole of MM' is evidently the other point of intersection of the nine-point circle and the joint-director circle of X and Y.

Hence, the nine-point circle and the joint-director circle of X and Y will touch each other when LL' and MM' coincide, *i.e.* when O lies on the conic Y. Thus Rogers' theorem is proved.

A. A. KRISHNASWAMI AYYANGAR.

975. [L¹. 19. d.] *Note on Note* 953 (*Gazette*, May 1930, p. 111), *A Generalization of Feuerbach's Theorem.*

For a euclidean plane, Neuberg* has defined the orthopole with respect to a triangle, ABC, of a line, l, as follows: Let L, M, N denote the orthogonal projections of A, B, C respectively on l; then the perpendiculars from L, M, N on BC, CA, AB respectively meet at a point, O, the orthopole of l with respect to ABC. O is the meet of the Wallace (Simson) lines of the common points of l and the circumcircle of ABC; if l is a diameter of that circle, it follows that O lies on the nine-point circle of ABC.

Let any conic, confocal with a conic inscribed to ABC, be termed *infocal* to that triangle; then the pairs of tangents to any infocal conic, perpendicular to BC, CA, AB meet those lines respectively at six points on a circle, which may be termed the *pedal circle*, with respect to ABC, of the conic. Professor Rogers's theorem may now be enunciated more exactly as follows: If an infocal conic passes through the circumcentre, then its pedal circle touches the nine-point circle, at the orthopole of that diameter of the circumcircle which touches the conic. This theorem is a case of the following: The pedal circle of an infocal conic meets the nine-point circle at the orthopoles of those diameters of the circumcircle which touch the conic; which may further be generalized as follows: If a variable infocal conic touches a fixed straight line, then the pedal circle of the conic remains orthogonal to a fixed circle, of which the centre is the orthopole of the given line. The last theorem is an extension of one given by T. Lemoyne†, in which isogonally conjugate point-pairs take the place of infocal conics.

Analogous theorems in a non-euclidean plane, and in higher space, have been given by the present writer‡; who has also‖ called attention to a generalization, for a euclidean plane, of Feuerbach's theorem which is closely related to those of this note.

J. P. Gabbatt.

976. [L¹. 10. a; L². 19. d.] *A Property of a Parabola inscribed in a Triangle.*

In the May issue of the *Gazette* (vol. xv. p. 111) Professor Rogers proved that the joint director circle of two confocal conics, one inscribed in a triangle, the other passing through the circumcentre, touches the nine-points circle.

His method of proof makes use of the constancy of the product of the focal perpendiculars on a tangent to a central conic and so is "ultimately applicable to parabolic cases."

The object of this note is the consideration of such cases from the purely geometrical standpoint.

It will be shown that:

(i) The circumcircle of the triangle formed by three tangents to a parabola touches the directrix of any parabola having the same focus and passing through the circumcentre.

(ii) (*a*) The locus of the intersection of orthogonal tangents one to each of a pair of confocal parabolas is the straight line midway between their directrices, and (*b*) this line touches the nine-points circle of any triangle whose sides touch one parabola and whose circumcentre lies on the other.

(i) If P is the circumcentre of such a triangle S lies on its circumcircle and, if PM is the perpendicular to the directrix of a parabola through P, $SP = PM$, which proves the result.

(ii) (*a*) If SY_1, SY_2 are the perpendiculars from the focus on the orthogonal tangents from T, Y_1 and Y_2 lie on the tangents to the parabolas at their vertices and so the mid-point of Y_1Y_2 lies on the line midway between these

* *Nouv. Correspondance math.* 2 (1876), 189.

† *Nouv. Ann.* (4) 4 (1904), 400-402.

‡ *Proc. Camb. Phil. Soc.* 22 (1925), 751-758.

‖ *Math. Gazette*, 12 (1925), 507.

tangents. But the mid-point of Y_1Y_2 is also the mid-point of ST. Hence T lies on the line midway between the directrices.

(*b*) Denote the parabolas by p_1, p_2, their directrices by d_1, d_2.

The orthocentre H of a triangle formed by three tangents to p_1 lies on d_1, and every line from H to the circumcircle is bisected by the nine-points circle. Hence, using (i), the locus (ii) (*a*), which is midway between H and d_2, touches the nine-points circle.

B. E. LAWRENCE.

977. [K. 2. a.] *An envelope associated with a triangle.*

I should like to insert this brief proof as correction of the statement at the end of a former Note: vol. xiv. p. 142.

Let α, β, γ, the complexes of the vertices, be the roots of the equation

$$z^3 - pz^2 + qz - r = 0,$$

and let $zw - \bar{z} = \mu$ be the transversal, $|w|$ being unity.

The equation of the side $\beta\gamma$ being $z + \bar{z}\beta\gamma = \beta + \gamma$, that of the perpendicular through its intersection with the transversal will be

$$z - \bar{z}\beta\gamma = p - \alpha + \frac{2r\mu}{\alpha - wr}.$$

The area contained by three such lines will be proportional to the *square* of the alternant in α, β, γ, sufficiently indicated by its first row.

$$\left| 1,\ \beta\gamma,\ p - \alpha + \frac{2r\mu}{\alpha - wr} \right|.$$

This has then one or other of two constant values.

$$\text{This alternant} = -|1,\ \beta\gamma,\ \alpha| + 2r\mu\,|1,\ \beta\gamma,\ \frac{1}{\alpha - wr}|;$$

$$\therefore\ \mu\,|1,\ \beta\gamma,\ \frac{1}{\alpha - wr}| = \frac{\mu}{\Pi(\alpha - wr)}\left|\alpha - wr,\ r - \beta\gamma\, wr,\ 1\right|$$

$$= \frac{\mu}{\Pi(\alpha - wr)}\left|\alpha,\ -\beta\gamma,\ wr,\ 1\right| \text{ and therefore } \frac{\mu wr}{\Pi(\alpha - wr)}$$

has one of two constant values, of which call either λ.

Hence the transversal equation is

$$zw - \bar{z} = \mu = \frac{\lambda}{wr}\,(r - wrq + w^2r^2p - w^3r^3).$$

The envelope is got by differentiating from w; the equation is

$$z = \lambda\left(-\frac{1}{w^2} + rp - 2wr^2\right)$$

and the envelope is therefore one of two hypocycloids.

L. J. ROGERS.

978. [A. 3. g.] *Ramanujan and Fürstenau's process.*

At the end of his very interesting lecture on Ramanujan *, Dr. Wilson offers for solution an example from Ramanujan's note-books which reads:

"To find convergents to a root of the equation

$$1 = A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \dots$$

If $$P_n = A_1P_{n-1} + A_2P_{n-2} + A_3P_{n-3} + \dots + A_{n-1}P_1$$

and $P_1 = 1$, then P_n/P_{n+1} approaches x when n becomes greater and greater."

* *Math. Gazette*, vol. xv, (p. 93, May 1930).

It is to be observed that this is the method which Fürstenau gave and proved rigorously in 1867. He also applied it to the case of repeated roots and other writers have since given independent proofs.

In effect, the method of procedure is as follows :

Multiply by powers of x and transpose : we get

$$0=-1+A_1x+A_2x^2+A_3x^3+\dots \qquad \dots\dots(1)$$
$$0=\qquad -x+A_1x^2+A_2x^3+\dots$$
$$0=\qquad\qquad -x^2+A_1x^3+\dots$$
$$\dots\dots\dots\dots$$

From this infinity of equations we select systems which have a finite number of terms, and in each system we treat the several powers of x as so many independent variables and solve for the first power only, as follows :

$$0=-1+A_1x. \qquad \text{Solution, } x=1/A_1, \quad (=x_1).$$

$$\begin{cases} 0=-1+A_1x+A_2x^2, \\ 0=\qquad -x+A_1x^2. \end{cases} \text{Solution, } x=A_1/(A_1^2+A_2), \quad (=x_2).$$

$$\begin{cases} 0=-1+A_1x+A_2x^2+A_3x^3, \\ 0=\qquad -x+A_1x^2+A_2x^3, \\ 0=\qquad\qquad -x^2+A_1x^3. \end{cases}$$

$$\text{Solution, } x=(A_1^2+A_2)/(A_1^3+2A_1A_2+A_3), (=x_3), \text{ etc.}$$

The sequence x_1, x_2, x_3, ... is that which is given by Ramanujan's rule and tends to the least root of equation (1), if it exists.

The general treatment, which of course involves the use of determinants, leads to Ramanujan's expression. A complete proof of Fürstenau's method is given in F. Riesz's *Les systèmes d'équations linéaires à une infinité d'inconnues* (Gauthier-Villars). V. NAYLOR.

[There is really no mystery about Ramanujan's discovery of this rule. In Carr's *Synopsis*, a book which he is known to have used, are given for the equation

$$x^n+p_1x^{n-1}+\dots+p_n=0$$

(i) the formulae (p. 139)

$$s_m+p_1s_{m-1}+p_2s_{m-2}+\dots+p_{m-1}s_1+mp_m=0, \quad m \leqslant n;$$
$$s_m+p_1s_{m-1}+p_2s_{m-2}+\dots+p_ns_{m-n}=0, \quad m>n;$$

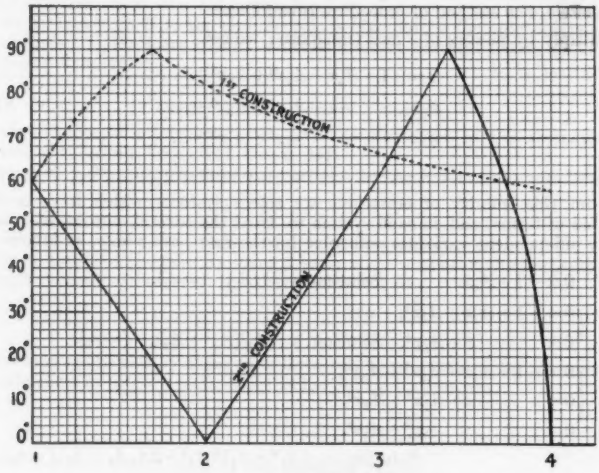
(ii) the theorem (p. 141) that s_{m+1}/s_m approximates in suitable circumstances to the numerically largest root. The modifications to the form in which Ramanujan propounded the rule are typical in spirit; in substance they seem trivial until we notice that it is the formula for the case in which m is greater than n that is invoked although n is to be infinite and m finite.

E. H. N.]

979. [K[1]. 21. a.] *Comparative Accuracy of two Constructions.*

Of the two very beautiful and interesting constructions given by Mr. Tuckey for the Mean Proportional between two given lines (*Gazette*, xiv, 542), the first (Fig. 3, page 543) seems to me much more accurate for most values of $OB:OA$ than the second (Fig. 4, page 544). Graphs of the acute

angles at which the circles cut show that in the first construction (dotted graph) the angle is never less than 58° for values of $OB : OA$ to which both



Graphs of the acute angles at which the circles cut in the two constructions.

constructions are applicable, while in the second (continuous graph) the angle fluctuates so wildly as to imperil the stability of the apple-cart.

W. HOPE-JONES.

980. [C. 1. e.] *Note on Taylor's Theorem.*

The following extension of Taylor's Theorem can hardly be new, but it does not appear in the well-known text-books.

Let $f(x)$ be differentiable n times; and let $P(x) \equiv Ax^n + \ldots$ be the polynomial of degree n having p-point contact with $f(x)$ when $x=a$, q-point contact when $x=b$, r-point contact when $x=c$, ..., where $p+q+r+\ldots=n+1$.

Consider the function

$$f(x) - P(x).$$

This vanishes $(n+1)$ times in the closed interval from the least to the greatest of the values $a, b, c, \ldots$. Hence, by an extension of Rolle's Theorem, its nth derivative vanishes for some value ξ of x which lies between the greatest and least of the values $a, b, c, \ldots$. Consequently

$$f^{(n)}(\xi) - An! = 0,$$

that is,

The coefficient of x^n *in the polynomial* $P(x)$ *has the form* $f^{(n)}(\xi)/n!$.

This includes Taylor's Theorem, which we get by putting $p=n$, $q=1$. For the polynomial $P(x)$ may then be written

$$P(x) = f(a) + (x-a)f'(a) + \ldots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + A(x-a)^n,$$

where A is determined by

$$f(b) = f(a) + (b-a)f'(a) + \ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + A(b-a)^n.$$

Ex. Given that $f^{(4)}(x)$ exists, prove the formula

$$f(b)=f(a)+\tfrac{1}{3}(b-a)\{f'(b)+2f'(a)\}+\tfrac{1}{6}(b-a)^2f''(a)-\tfrac{1}{72}(b-a)^4f^{(4)}(\xi).$$

Proof. Let $P(x)$ be a quartic having three-point contact with $f(x)$ when $x=a$, two-point contact when $x=b$. Then $P(x)$ may be written

$$P(x)=f(a)+(x-a)f'(a)+\tfrac{1}{2}(x-a)^2f''(a)+(Ax+B)(x-a)^3,$$

where A, B (only A is needed) are determined by

$$f(b)=f(a)+(b-a)f'(a)+\tfrac{1}{2}(b-a)^2f''(a)+(Ab+B)(b-a)^3, \quad \ldots\ldots\ldots\text{(i)}$$

$$f'(b)=f'(a)+(b-a)f''(a)+3(Ab+B)(b-a)^2+A(b-a)^3. \quad \ldots\ldots\ldots\text{(ii)}$$

Multiply (i) by 3, (ii) by $(b-a)$, subtract, and replace A by $f^{(4)}(\xi)/4!$, and we get the required formula.

Other such examples may be constructed *ad lib.* (see *e.g.* Hardy: *Pure Mathematics*, (1925), p. 300, Exs. 4, 5, 6, 7). F. BOWMAN.

981. [V. 1. a.] *Note on Abbreviated Symbols.*

Examining bodies and authors show neither that consistency nor that agreement among themselves in the use of abbreviations which the student of science expects; thus, he finds "grms", "gms", "grammes", "grams", "gm", "grm" used by various authorities for the same concept.

As a general rule, the letters retained in any abbreviation should be both just necessary and just sufficient for the purpose of uniquely symbolizing the concept to which they refer. In this way we remove all from the above list except "gm" and "gms." It is the object of this note to suggest that the singular form "gm" be invariably used; *i.e.* that we write "2 gm" and not "2 gms"; similarly, that we should use 2 lb, 2 ml, 2 yd, 2 min, in preference to 2 lbs, 2 mls, 2 yds, 2 mins.

There are sound scientific reasons in favour of this practice; for example, those who use operations with magnitudes in Mechanics could not justify the cancelling of "mls" with "ml" in such an expression as

$$2\ \text{mls} \times \frac{5280\ \text{ft}}{1\ \text{ml}} = 2 \times 5280\ \text{ft}.$$

In his *Operazione sulle Grandezze*, Professor Peano puts the matter clearly. He quotes Heron's definition: "Magnitude is that which can be increased or diminished", *i.e.* magnitude is that which can be "multiplied" by a numerical quantity.

Thus, an expression like "2 ml", "2 × ml" or simply "2 ml" may be called the "multiplication" of numerical quantity, (2), by the magnitude, (ml). The use of the word "multiplication" is justified because the operation associated with the dot obeys all the laws of Arithmetic multiplication. There is then no question about the final "s."

It is probable that those authors, who use an expression like "2 mls", have at the back of their minds some such ordinary operation as that of placing end to end two separate "mile" rods. Any visual aid of this type must break down when they come to write "1 ml" = "ml." Whatever physical picture we use in the initial stages, we are sooner or later forced to regard "ml" as an operator by means of which we make a certain length correspond with the abstract number "1".

Whether we use ordinary reasoning or abstract reasoning, the symbols we use should be scientifically consistent.

Peano uses the single letter G for gm, and C for cm.

We in England, who have to distinguish between "mile" and "metre", must use "ml" for "mile", if we use "m" for "metre".

The corresponding divergence of usage puts in an appearance in everyday language when the plural of a word indicating measure is in question.

On one hand, some authors who are competent to criticize English usage say, that certain words which indicate measure should have the same

form in the plural as in the singular. Accordingly, we should speak of a "2 mile walk", and not a "2 miles walk"; while in answer to the question "How far is A from B?" we should answer "2 mile" and not "2 miles".

On the other hand, many allow, and common practice indicates some latitude in the plurals of other words denoting measure; thus, we speak of a "2 foot rule", a "3 foot table", but in answer to such a question as "How long is this?" the answer might equally well be "3 foot" or "3 feet".

But here, again, authorities on syntax differ.

This difference they could remove by adopting the suggestion made to teachers of physical science at the opening of this note, viz. to use the singular form invariably, whenever a simple expression of magnitude is intended.

This they could justify by interpreting such an expression as "2 foot" as meaning "two times 1 foot", or "two times a foot".

H.M. Dockyard School, Devonport. V. NAYLOR.

Since writing the above note, I have referred to Kaye and Laby's *Physical Constants*, where, I take it, current practice among Physicists may be found.

Among others, the following expressions occur: (1) "contains 10 lbs. weight"; (2) "c.c."; (3) "100 c.cs."; (4) "wt of 1 gm × 1 cm = g dyne cms"; (5) "33,000 ft.-lbs. per min." In these, the parts of questionable significance are, (1) "contains", the "s", the "dot"; (2) each "dot"; (3) the "s" and the transposition of the "dot"; (4) the omission of a "dot", the "s" (5) the second "dot", the "dash", the "s".

Those who use operations with magnitudes would, of course, favour

(1) "10 lb wt"; (2) "cm^3"; (3) "100 cm^3";

(4) "g dyne . cm"; (5) "33,000 $\frac{\text{ft. lb wt}}{\text{min}}$". V. N.

982. [B. 1. b; N^2. 1. a.] *A Problem on Normal Congruences.*

Mr. L. E. Prior's note, 935, *Gazette*, xiv, p. 458 (May, 1929), is welcome as a contribution towards elegance of treatment in that somewhat ponderous subject Differential Geometry of Surfaces. I employed similar methods in the following property of normal congruences which I proposed as a problem in an examination for the London M.Sc. Degree in Dec. 1924.

"A congruence of rays of the type $(\xi - x)/X = (\eta - y)/Y = (\zeta - z)/Z$ is incident on a reflecting surface. Prove that the following expressions are invariant in the incident and reflected congruences, viz.:

$$\frac{e\beta^2 - 2f\beta a + ga^2}{v^2}, \quad \frac{ec\beta - f(b\beta + ca) + gab}{v^2}, \quad \frac{ac - b\beta}{v},$$

where $a = \Sigma x_1 X_1$, $b = \Sigma x_2 X_1$, $\beta = \Sigma x_1 X_2$, $c = \Sigma x_2 X_2$, $e = \Sigma X_1^2$, $f = \Sigma X_1 X_2$, $g = \Sigma X_2^2$, $v^2 = eg - f^2$ and suffixes denote differentiations with respect to the independent parameters p and q."

The notation here adopted is with a slight modification that of Forsyth's *Differential Geometry*; dashes will refer to the reflected congruence.

Since the incident and reflected rays are coplanar with the normal we have

$$\begin{vmatrix} X, & Y, & Z \\ X', & Y', & Z' \\ X'', & Y'', & Z'' \end{vmatrix} = 0,$$

where (X'', Y'', Z'') are the direction cosines of the normal to the reflecting surface.

Hence (following Forsyth, § 278) quantities λ and μ exist such that

$$X = \lambda X'' + \mu X', \quad Y = \lambda Y'' + \mu Y', \quad Z = \lambda Z'' + \mu Z', \quad \ldots\ldots\ldots\ldots (1)$$

and it follows easily as in that article that μ is the constant index of refraction, when there is refraction, and is -1 when there is reflection; as in the case contemplated.

We have then from (1), putting $\mu = -1$,

$$\left.\begin{aligned} \Sigma X x_1 &= -\Sigma X' x_1 \\ \Sigma X x_2 &= -\Sigma X' x_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Substituting for vX, $Y_1Z_2 - Y_2Z_1$ (Forsyth, § 162) and similarly for vY and vZ, and correspondingly for the reflected congruence, we have

$$\frac{1}{v}\begin{vmatrix} x_1 & y_1 & z_1 \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = -\frac{1}{v'}\begin{vmatrix} x_1 & y_1 & z_1 \\ X_1' & Y_1' & Z_1' \\ X_2' & Y_2' & Z_2' \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and

$$\frac{1}{v}\begin{vmatrix} x_2 & y_2 & z_2 \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = -\frac{1}{v'}\begin{vmatrix} x_2 & y_2 & z_2 \\ X_1' & Y_1' & Z_1' \\ X_2' & Y_2' & Z_2' \end{vmatrix}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Squaring both sides of (3),

$$\frac{1}{v^2}\begin{vmatrix} E & a & \beta \\ a & e & f \\ \beta & f & g \end{vmatrix} = \frac{1}{v'^2}\begin{vmatrix} E & a' & \beta' \\ a' & e' & f' \\ \beta' & f' & g' \end{vmatrix}$$

whence as $eg - f^2 = v^2$, $e'g' - f'^2 = v'^2$, E drops out from each side and we get

$$\frac{e\beta^2 - 2fa\beta + ga^2}{v^2} = \frac{e'\beta'^2 - 2f'a'\beta' + g'a'^2}{v'^2}.$$

Similarly, squaring both sides of (4),

$$\frac{1}{v^2}\begin{vmatrix} G & b & c \\ b & e & f \\ c & f & g \end{vmatrix} = \frac{1}{v'^2}\begin{vmatrix} G & b' & c' \\ b' & e' & f' \\ c' & f' & g' \end{vmatrix}$$

giving by expansion,

$$\frac{ec^2 - 2fbc + gb^2}{v^2} = \text{corresponding expression with dashed letters.}$$

Again taking the product of (3) and (4) and making appropriate substitutions

$$\frac{1}{v^2}\begin{vmatrix} F & a & \beta \\ b & e & f \\ c & f & g \end{vmatrix} = \frac{1}{v'^2}\begin{vmatrix} F & a' & \beta' \\ b' & e' & f' \\ c' & f' & g' \end{vmatrix}$$

i.e. (F cancelling out)

$$\frac{ec\beta - f(b\beta + ca) + gab}{v^2} \text{ is invariant.}$$

The invariance of the third expression is easily deduced from those already found.

Forsyth uses the device of squaring a determinant to express its value in terms of known quantities more than once in the work above referred to, *e.g.* in his chapter on twisted curves and again in his analysis of ruled surfaces.

July 5, 1929. E. H. SMART.

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794. *Tables and Formulae for the Computation of Life Contingencies.* By Peter Gray. The distinct characteristic of this work is the introduction of Gauss's Table into the calculation of life contingencies; and by this one introduction the name of the author will live in the history of the subject. [Gauss's table] enables anyone to find the logarithm of a sum or difference by knowing the logarithms only of the components.—*Athenaeum*, 1849, p. 380.

REVIEWS.

The Rhind Mathematical Papyrus. By A. B. CHACE, L. BULL, and H. P. MANNING. Vol. I 1927; Vol. II 1929. $20. (Mathematical Association of America.)

Dr. Chace has no reason to regret the fifteen years of his life which he has devoted to the preparation of this great work, for it is a noble contribution both to Mathematics and to Egyptology. The fact that it reproduces photographically the whole of the original would in itself give it a claim to rank as the classical edition of the papyrus. Its authors have further attained an astonishing standard of accuracy, and that without showing any tendency to sneer at the only too numerous inaccuracies of earlier editors. Dr. Chace has refused to be dictated to by any considerations of space, and consequently his work, especially the second volume, avoids that cramping which is apt to be so disturbing and confusing in a work of this kind. The numbering of the plates and the system of cross-references in the volume which contains them are admirable.

The work is intended in the main for mathematicians rather than Egyptologists, and the hieroglyphic transcription is therefore accompanied by a transliteration into sounds—written backwards in order to keep in exact touch with the hieroglyphs—a very literal translation, and a number of grammatical and syntactical notes, mostly elementary in character. It is not easy for the philologist to put himself into the position of the mathematician and to realize exactly what is the value to the latter of this *apparatus criticus*. Obviously the mathematician cannot really check the philologist's translation; at the same time one can see how the material provided does help him to understand just how that translation is obtained, gives him a deeper insight into the Egyptian mathematician's mode of expression, and hence of thought, and even enables him in some cases to test the possibility of an alternative rendering of his own suggested by his interpretation of the mathematical data. That this is really so seems proved by the fact that nearly every mathematician who has made valuable contributions to Egyptian mathematics has found it necessary to form some sort of acquaintance with the Egyptian text.

This literal translation and the notes which accompany it are very well done, and there are few things with which fault could be found. In Problem 21, line 1, however, the word *m* is surely the interrogative pronoun, "What completes . . . ?", rather than the particle used after the imperative, which does not seem to occur in this papyrus. In Problem 50 *'ḥt dbn* cannot possibly mean "a round field", since *'ḥt* with the plural strokes can only mean "land", and *dbn*, in view of its determinatives, must be a noun. Translate "a circular piece of land", literally "land, a circle"—a common Egyptian idiom. Problem 62, note 7, involves a curious misconception. The word *n* does not stand between two nouns, for the *ntt* which precedes it is the relative pronoun. Whether we translate "amount of" or "assignable to" it is certain that *n* is the preposition, not the genitival adjective.

In the actual interpretation of the problems Dr. Chace would himself be the first to admit that he has not very much that is new to tell us; so much has been written about this aspect of the papyrus since Eisenlohr's clever edition in 1877 that there is probably very little more to be said. This has of late been generally recognized, and the interest of writers has consequently moved over from the problems themselves to the nature of the mathematical conceptions and processes involved in them. On this subject Dr. Chace and his collaborators have much to say and some new points to make.

Dr. Chace points out that division in the Egyptian sense may be regarded as a second type of multiplication, in which the multiplicand and product are given and the multiplier is to be found. This gave the Egyptians no trouble. Trouble did, however, arise when the multiplier and product were given and the multiplicand had to be found, for the question of units had now to be faced. The multiplier might, for instance, be the number 3 and the product 15 loaves. Now no amount of "operating on 3", as the Egyptian called his multiplication process, will produce 15 loaves; it can only be done by

"operating on" 3 loaves. The Egyptian fully realized this, and, according to Chace, either treated his multiplier as multiplicand, thus entirely throwing over his units for the moment, to restore them at the end by adding the word "loaves", or whatever it might be, to the resulting 5, or else, anxious to retain his units and thus to proceed logically throughout, used the method of "false position" exemplified in Problems 24 to 27.

Dr. Chace is among those who deny to the Egyptians the conception of a general fraction, *i.e.* one whose numerator is greater than unity. Is it not possible that the variance of opinion among mathematicians on this point lies in the ambiguity of the word "conception"? That the Egyptians had no notation for such fractions (except $\frac{2}{3}$) is certain. On the other hand, they had a clear pictorial conception of say $\frac{1}{7}$ as the result of cutting up a unit into seven equal parts, and this pictorial conception must have included all possible combinations of these seven separate parts, not only the $\frac{6}{7}$ which was left when one was taken away, but also $\frac{2}{7}$, $\frac{3}{7}$, etc. What is more, in actually working with fractions they were accustomed to double them; if when they came to double $\frac{1}{7}$ they wrote $\frac{1}{4}+\frac{1}{28}$ and not $\frac{2}{7}$, this is a matter of notation and method, not of conception, since it was for them, as it is for us, far easier to conceive pictorially an object divided into 7 parts and two of these taken together than to conceive an object first divided into 4 parts, and one taken, and then divided into 28 parts and one again taken. In the same direction points the fact that when two fractions with the same even denominator, *e.g.* 14, come together in the working of a sum they are at once reduced to a single fraction, in this case $\frac{1}{7}$. If the worker had no conception of $\frac{2}{14}$, the two $\frac{1}{14}$s would have to remain side by side in the reckoning and could never be united.

Dr. Chace disputes Gunn's statement that the Egyptian "like everyone else had ultimately no way of arriving at $\frac{2}{3}$ but via $\frac{1}{3}$". To his argument on this point we may add that, if two-thirds of a number was envisaged as that quantity which when its half is added to it produces the number in question, then $\frac{2}{3}$ could be reached directly through a half, without having recourse to the less elementary idea of a third.

In the matter of the addition of fractions Dr. Chace is loth to admit that the Egyptian procedure is really identical with our process of reducing to a common denominator. He believes that what the Egyptian does is to "apply" the fractional expression which he wishes to simplify, say $\frac{1}{5}+\frac{1}{6}+\frac{1}{10}+\frac{1}{30}$, to a "particular number", say 30, and, finding that when applied to this number it gives half of it (*i.e.* 15), to assume that it will give half of whatever number it is applied to, and that it is therefore identically equal to $\frac{1}{2}$. Dr. Chace resolutely denies that this process involves any suggestion that $\frac{1}{5}$ is equal to $\frac{6}{30}$, and so on. And no doubt there are many who will agree with him.

There are some good comments on the division of 2 by odd numbers, though this treatment as a whole must be regarded as superseded by Vogel's masterly discussion in his *Die Grundlagen der ägyptischen Arithmetik*, Munich, 1929, which appeared after Dr. Chace's book went to press.

It is a little disappointing, at any rate to the Egyptologist, to find that Dr. Chace still denies to the Egyptians the merit of having discovered the correct formula for the area of the triangle in general. It is not possible logically to demonstrate that the word *mryt* in the Rhind problem means the vertical height (as against side), or that the triangle there dealt with is intended to be scalene; but in the face of the new evidence from the Moscow papyrus (see Gunn and Peet in *Journal of Egyptian Archaeology*, xv, 1929, pp. 171-6) it seems difficult to doubt any longer that the Egyptian solution was correct and applied to all triangles whatsoever.

The suggestion that the area of the circle was got from the volume of the cylinder experimentally, by measuring the base of a square prism which was of equal height with the cylinder and held the same quantity of water, is ingenious; but the fact that in the papyrus the problems on volume precede those on the area of the circle is hardly to be used as an argument in its favour, for the arrangement of the papyrus is almost haphazard, and it may consist of sections chosen almost at random from separate treatises.

One point remains. Dr. Chace is opposed to the school of thought which regards Egyptian mathematics as purely practical in scope. It is possible that this school, of which I am an adherent, has overstated its case. We are all ready to admit that to achieve any practical result in mathematics there must be a certain theoretical interest in the subject. What the Egyptians did not show, however, was an interest in the subject for its own sake, and a tendency to develop such branches of it as could not be given any practical application. Dr. Chace's contention that certain problems involve figures so fantastic that they must be regarded as theoretical problems given in concrete form is met by Gunn's brilliant suggestion (Gunn and Peet, *op. cit.* pp. 184-5) that the problems in these mathematical handbooks are to be regarded not as problems containing formulae but as formulae to be applied to the solution of problems. In any case, in a country which built as many pyramids as Egypt the occurrence of a problem determining the volume of a truncated pyramid hardly seems to me evidence for a theoretical interest in mathematics (p. 43, n. 1); nor again does the interest taken by the surgeon of the Edwin Smith Medical Papyrus in cases which he admitted to be beyond treatment prove any other than a purely practical interest in surgery and medicine (*ibid.*). Still, the case on the other side may easily be overstated, and Wieleitner's *War die Wissenschaft der alten Aegypter wirklich nur praktisch* is doubtless a useful corrective to the views of those of us who, from long contact with the ancient Egyptians, have come to regard them as an intensely practical race with little or no taste for speculation.

The value of the work is very greatly increased by the incorporation in it of Professor Archibald's Bibliography of Egyptian Mathematics, which some of us have been privileged to use in typewritten form for some time past. This is a piece of bibliographical work of the very highest order, and it is to be hoped that some means may be found of ensuring the printing from time to time of such additions to it as may become necessary.

T. Eric Peet.

Differential Equations. By F. R. Moulton. Pp. xv + 395. 24s. 1930. (The Macmillan Company.)

Even the humblest author cannot survey the whole vast domain of differential equations in a volume of less than four hundred pages; the more eminent the writer, the more eclectic his view-point. And so it is a little unjust to issue this book under the bald and over comprehensive title which it bears. Something a little more specific would have been fairer, as "General Classes of Ordinary Differential Equations, and Other Matters of Interest to the Astronomer". Partial Differential Equations are wholly excluded, and in the domain of ordinary equations, to quote the author's preface, "it has not been possible to include a discussion of the existence and the properties of solutions at, and in the neighbourhood of, singular points of differential equations; nor has it been possible to include solutions subject to general boundary conditions rather than to initial conditions".

To many of us these things are the wine of the subject; as they are excluded from the banquet before us, the eighteen courses served up for our consumption lie heavily and lack relief,—the more so as they show the hand of the good solid plain cook rather than the delicate touch of the *chef de cuisine*.

To proceed to detail, Chapter I is headed *Generalities and Definitions*. A knowledge of the elementary properties of differential equations, and of the classical devices for cracking the few not-too-hard nuts among them, being assumed, the reader is introduced to the normal system written in the compact notation:

$$\frac{dx_i}{dt} = f_i(x_j;\ t) \quad (i, j = 1, 2, \dots, n). \tag{1}$$

Attention is throughout the work focussed upon four essential points: (1) the formulation of specific problems; (2) proofs of the existence of solutions; (3) properties of the solutions; (4) practical means of finding the solutions. By a solution is meant an explicit determination

$$x_i = \phi_i(t)$$

of the dependent variables subject to specified initial conditions

$$x_i(t_0) = a_i, \tag{2}$$

as distinct from an integral, which is a function $F(x_1, \dots, x_n; t)$ such that

$$F(\phi_1, \dots, \phi_n; t) = C,$$

for every solution $\phi_1, \dots, \phi_n$, where C is a constant dependent upon $t_0, a_1, \dots, a_n$.

Chapter II, *Analytic Differential Equations*, has two main subdivisions. In the first, assuming that the functions f_i are analytic in a domain about the initial point $(t_0, a_1, \dots, a_n)$, the existence in the neighbourhood of that point of a unique analytic solution of the system (1, 2) is proved by the *calcul des limites* of Cauchy. In the second, the properties of the solution are discussed; in particular they are proved to be analytic functions of the initial point. Direct extension is made to the case in which the f_i involve a parameter μ.

Chapter III, *Differential Equations Analytic in a Parameter*, leads up to a discussion of a more general system in which the right-hand members are merely continuous in t, but analytic in the x_i and in the parameter μ. By methods perfected by Poincaré the formal solution is developed as a power-series in μ, and convergence in a sufficiently-restricted domain is established.

Chapter IV is headed *Variation of Parameters*. As Moulton remarks, it is convenient to retain this historic title, although the present treatment is extremely general, and "is really a transformation of variables in which the equations of transformation are so chosen as to simplify the results to the greatest possible extent". Practically, it may be desirable to repeat the process many times. It is related that Delaunay, in his lunar theory, after reducing the fundamental equations to the most advantageous form, made 57 complete, and 440 partial transformations, work which required twenty years of arduous labour.

Chapter V, *Integrals of Ordinary Differential Equations*, proves that the system (1, 2) admits of a fundamental set of n independent integrals. The differential equations of the problem of three bodies, being equivalent to a normal system of the eighteenth order, possess eighteen integrals. Ten of these are classical and have simple physical interpretations. The problem of choosing uniformising variables for the remaining integrals has been solved only in special cases.

Chapter VI, *Analytic Implicit Functions*, seems foreign matter, but the methods of the theory very closely resemble those of Chapters II and III. Moreover, implicit functions are very frequently involved in the manipulation of differential equations, as, for example, in the depression of the order of a system when certain integrals are known.

Chapter VII, *The Problem of Elliptic Motion*, illustrates the theory which has been developed in the preceding chapters.

Chapter VIII, *The Sine-Amplitude Function*, develops the main properties of the Jacobian elliptic functions (though Jacobi is only once, and then somewhat casually, mentioned) from the differential equation. It serves as a further illustration of the theory; otherwise it contains little of interest, and elliptic functions are not explicitly used in subsequent work. Moulton uses the notation $sn(t, k^2)$ and breaks with honourable tradition in denoting the periods by P_1 and $\sqrt{-1}P_2$. It may be noted in passing that imaginary numbers, even when they occur as exponents, are always written in this clumsy form.

Chapter IX, *The Deviations of Falling Bodies*, occurs opportunely here since differential equations and implicit functions are both involved. The deviation in longitude of a falling body is eastward from a plumb line suspended from its point of fall. Gauss found its amount to be small but significant, for example, 3·6 inches for a fall of 1000 feet in latitude 40°. The deviation in latitude presents greater difficulties, and early estimates were erroneous. The first correct result was obtained by Roever in 1911 and afterwards confirmed by Moulton himself. The deviation is always equatorwards, but insignificant, amounting to less than a thousandth of an inch in the above case.

Chapter X, *The Damped Gyroscope*, illustrates the advantages which accrue from a suitable choice of the transformations to be applied to the Eulerian equations.

Chapter XI, *The Method of Successive Integrations*, perhaps more generally known as that of Successive Approximations, demands less stringent restrictions on the $f_i(x_j : t)$ than do the methods of Chapters II and III. An important modification of the procedure lies in replacing the usual first integration

$$x_i^{(1)} = a_i + \int_{t_0}^{t} f_i(a_j ; t)\, dt$$

by the more general

$$x_i^{(1)} = a_i + \int_{t_0}^{t} f_i(a_j + \phi_j ; t)\, dt,$$

where the ϕ_j are functions of t, vanishing for $t = t_0$ and otherwise suitably conditioned.

In Chapter XII, *Numerical Solution of Differential Equations*, this modification becomes of practical importance. This numerical adaptation of the method of successive integrations was previously published in the author's *New Methods in External Ballistics* (1926).

Chapter XIII, *The Cauchy-Lipschitz Process*, deals with a method historically interesting and essentially simple. The method is, however, not readily adaptable to numerical work.

Chapter XIV, *Generalities on Linear Differential Equations*, introduces a single general idea, the fundamental set ; it is mainly concerned with *Equations for Variations*. Suppose that for special values of the initial conditions (2), the system (1) has a simple solution. Then the equations which determine the variation of the solution due to small changes in the initial values are linear, and thus the question of the stability or instability of the solution presents no essential difficulty.

Chapter XV, *Linear Differential Equations with Constant Coefficients*, deals with the homogeneous system of order n. Any discussion of this case in all its generality is apt to be excessively tedious compared with the essential simplicity of the case of a single equation of order n. The method here followed is that of Nyswander (1925), which is quite the neatest and simplest available. No mention whatever is made of the fundamental work of Chrystal.

Chapter XVI, *Applications of Linear Differential Equations*, ranges from the stability of the planetary orbits to the radio carrier wave.

Chapter XVII, *Linear Differential Equations with Periodic Coefficients*, builds up a theory parallel to that developed in Chapter XV. The most interesting, and most important, feature, namely the possible existence of auto-periodic solutions, is untouched, and it is most surprising that the name of Mathieu is not even mentioned.

Chapter XVIII, *Differential Equations in Infinitely Many Variables*, touches on a branch of analysis to which very little attention has been given. The postulation of an infinite universe has many curious and perplexing implications which have no parallel in the finite case. For instance, the ten classical integrals exist formally in an infinite universe but are divergent. Moulton's own work forms the basis of this chapter, which contains some hitherto unpublished matter.

Certain standard theorems of analysis which were referred to in the text are proved in appendices.

It will thus be seen that, when applications to physical problems are set aside, the treatise is concerned almost entirely with existence theorems of considerable generality. A high standard of rigour has been maintained throughout, but the very generality and logical completeness of the exposition diminish its attractiveness. Nevertheless we are grateful to the author for having collected the fundamentals of the science into one volume. For accounts of what is built thereon we must look elsewhere.

A mathematician is known by the authorities he cites ; Moulton ignores entirely such men as Bessel, Fuchs and Sturm, and merely nods to Gauss and Legendre as he meets them in bypaths. In recent work the point of view is strongly transatlantic and references to American writers outnumber all other references in the ratio of five to one.

As indexing is such a dull and tiresome business, we may perhaps excuse the author the inadequacy of his index and its occasional inaccuracy, but no excuse whatever can be made for the lack of care which is evident in the references. To quote the *Annalen* again and again when *Crelle* is meant is quite unpardonable. Nor should we have expected to meet such barbarisms as *Acta Société Scientifique de Fennicae* and *Rendiconti Matematico di Palermo*.

To a work which would have tended to revive in English universities the almost extinct spark of interest in differential equations we should have extended a most cordial welcome. In America during the last thirty years warm enthusiasm has brought achievement in this field. What Americans have done in that period is insufficiently well known over here, and, were it presented to us in connected form, would make a valuable addition to our literature. Unfortunately the work under review is so restricted in scope that the most stimulating of recent discoveries are not included. Nor does it show that brilliance of exposition which inspires others to original work.

Within its self-imposed limits, however, this work is without any doubt of very great value for the fundamental information which it contains. But it must be remembered that the most remarkable of architectural features are not always found in basements. E. L. INCE.

Analytische Geometrie. By L. BIEBERBACH. Pp. iv + 120. Rm. 6.60. 1930. Teubners Mathematische Leitfaden, 29. (Teubner.)

On reading this little book I remember with satisfaction that some time ago I wrote in a *Gazette* review, "The mathematician is not going to say that when he calls himself a geometer . . . he necessarily accepts a standard of reasoning which as an analyst he regards as unworthy of confidence", and again, "Sooner or later the student must learn that geometry is a rigorous branch of mathematics". Prof. Bieberbach is known to us best as an analyst, and here he handles the elements of analytical geometry with the logical precision which we recognise nowadays to be indispensable in teaching the elements of the theory of sequences. This is, for example, one of the very few books in which there is an explicit recognition of the distinction between the zero vector and the zero scalar. From a basis of axioms concerning vectors and points, with chapters on determinants and matrices, Prof. Bieberbach reaches chapters on the metrical classification of surfaces of the second order, on the affine classification of curves of the second order, and on the geometrical properties of affine transformations. It is an astonishing range for a tiny volume, and yet there is room for examples to show how a clear grasp of principles involves facility in practice.

In brief, this is a book to be recommended warmly to every student of the subject, and also, let me add, to everyone who wishes to make a start in reading mathematics in German, for the material is varied and not abstruse, and the style is straightforward. E. H. N.

Barlow's Tables of squares, cubes, square roots, cube roots, and reciprocals, of all integer numbers up to 10,000. Third edition. Edited by L. J. COMRIE. Pp. xii + 208. 7s. 6d. 1930. (Spon.)

Nothing more was requisite for the preparation of his *New Mathematical Tables*, said Peter Barlow modestly in his Preface in 1814, than "a moderate skill in computation and a persevering industry and attention; which are not precisely the qualifications a mathematician is most anxious to be thought to possess". And "being well aware that the utility of the performance depended wholly upon its accuracy", Barlow took immense care both in the computations and in the printing. As a reward, 'Barlow' is one of the minor classics in the mathematician's reference library. In 1840, the original edition being out of print, De Morgan designed a new edition of the table of squares, cubes, square roots, cube roots, and reciprocals, which, with a column of factors, formed just half of Barlow's own work. This edition, supervised with the utmost care and stereotyped before publication, has been reprinted again and again, until now the plates show signs of wear; the ever-increasing demand "has made it possible for the publishers to face the expense of

resetting the work", and a notable task has been accomplished under the skilful guidance of Dr. Comrie.

The new editor has added a column giving the square root of $10n$ throughout the whole range of De Morgan's table, he has printed the first differences of the square roots, cube roots, and reciprocals, and he has explained in a brief introduction the simplest methods of interpolation, drawing attention to the processes that become practicable if a calculating machine is available. Although these substantial improvements are not beyond the scope of the 1840 title, Dr. Comrie does himself an injustice, for there are other tables in his book. Powers from n^{11} to n^{20} up to 10, $n!$ up to 100, and $1/\sqrt{n}$ up to 1000, appear in 'Barlow' for the first time; powers from n^5 to n^{10} up to 100, and n^4 up to 1000, are restored.

The omissions from the original edition are now: factors up to 10,000; the fifth power up to 1000; a table of $y^3 - y$ for the solution of the irreducible case in cubic equations; primes under 100,000; hyperbolic logarithms from 1000 to 10,000; the first six binomial coefficients for fractional values of n; a table giving the decimal equivalent and the logarithm of certain common coefficients; an extensive synopsis of formulae, ending with a table of integrals ('fluxions and fluents') and the elements of the solar system; and tables of weights and measures and of specific gravities ("platina, putrified, 19·5000; egg of a hen, 1·090; ewe's milk, 1·0405; woman's milk, 1·0203").

It must not be supposed that Dr. Comrie has been content to reprint without examination. Many of the entries have been recalculated, and all have been verified scrupulously in some way or other. De Morgan in his preface remarked that "it must take years of trial and use of the tables to establish their freedom from error", and indeed in spite of all care a systematic error did affect the cube roots from 9600 to 9800 in the 1840 edition. Dr. Comrie, with human fallibility kept in check by mechanical means unknown to De Morgan, could speak less diffidently if he wished, and with the confidence of a Halley who cannot live to see the outcome, I venture to predict for his edition a useful life no shorter than the ninety years of its predecessor.

E. H. N.

Foundations of Geometry and Induction. By JEAN NICOD. With Prefaces by Bertrand Russell and André Lalande. Pp. 286. 16s. 1930. (Kegan Paul, Trench, Trubner).

Nicod is probably best known in this country by his article on the primitive propositions of Logic published in the *Proceedings* of the Cambridge Philosophical Society during the war, in which he showed that one formal axiom, together with certain rules of operation, constitutes a sufficient basis for the logic of elementary propositions. That article and the book under review are his most important works. He died in 1924.

In the essay on the "Geometry in the Perceived World", which is the longer and more important of the two in this book, Nicod deals with the relation between abstract geometry built up from axioms and definitions, and the geometry derived from experience. Now geometrical notions are, or could be, derived from various types of human experience; from visual sensations, from tactile sensations, from sensations of movement. Nicod takes one type of experience at a time, in a simplified form. He first considers a creature endowed only with a sense of hearing, who hears sounds as he moves about, and knows whether one sound is like another, and whether it came before or after the other. From these simple experiences can be constructed a geometry analogous to one-dimensional *analysis situs*.

Much more interesting is the next hypothesis, in which a creature is imagined whose sole perceptions consist of sensations of total movements, so that when he is rigidly transported from a position A to a position B, he feels the movement as a whole not made up of parts, and knows when one movement is like another. The creature starts from an initial position, and knows when he returns to that position. From this unpromising material, Nicod shews how to build up a complete euclidean geometry. (Certain assumptions will of course be made, but these will be experimental facts for our creature). In constructing geometry on this basis, Nicod uses results obtained in work on

the axioms of the geometry of displacements, and it is most interesting to see these abstract investigations clothed in the flesh and blood of experience.

Succeeding chapters deal with the geometry of visual perceptions, first under the conditions that all bodies in space are stationary and all visible together, and secondly under assumptions which correspond to the fact that our visual world is—in a certain sense—two dimensional.

Finally a creature is imagined more like ourselves, endowed both with kinesthetic and visual sensations; Nicod considers how such a being could melt into one the various geometries given by his various experiences, and so reach some sort of notion of a unitary external space.

It is obvious that this essay is of very great importance; it carries out with extreme ingenuity and critical skill some of the programme left unfinished by Poincaré.

The second essay on "the logical Problem of Induction" seems of less importance; it consists largely of criticisms of the work of Keynes. The conclusions reached are, that induction by simple enumeration is a fundamental method of proof, and does not need the hypothesis of determinism for its validity, and that the probability of a hypothesis is increased even when a new confirmative fact merely repeats an old fact, but that no one has yet proved that by induction the probability of a law can be indefinitely increased.

Bertrand Russell contributes a preface to the first essay, and the influence of his teaching is manifest throughout the book. Lalande's preface to the second essay contains an interesting biographical sketch of Nicod.

The translation has been well done, as far as the English style is concerned, but there are a good many misprints, and sometimes the technical discussion is not clear. For instance, the last two lines on p. 119 seem to be a literal and unintelligible translation, (I have not access to the French original), and surely "without translation" on p. 173 line 3 f.b. should be "without rotation". But these defects will not hamper very much the reader who follows the argument. H. G. F.

Preliminary Mathematics for Engineers. By W. S. Ibbetson. Pp. xi + 152. 3s. 6d. 1930. (Pitmans.)

The book is intended as a first year course in mathematics for students of engineering—presumably evening students. The author opens his Preface by the remark that "The great stumbling block to the progress of evening students is their lack of *suitable* mathematical knowledge."

Nearly half the book is devoted to arithmetic, including simple graphs. Vulgar and decimal fractions are treated at length, and there is a chapter on "Ratio, Proportion, and Percentages." The rules for the various operations are baldly stated, *e.g.* "Invert the divisor and multiply the two fractions together," and then examples are given, with marginal comments at each step. This is probably justifiable, as most of this work *should* be nothing more than a revision of school arithmetic. The section on graphs, though short, does not stop at drawing them, but endeavours to give some idea of their uses, and in the exercises, "reading" the graph in some way is usually required.

The algebra commences with the old-fashioned sequence of explanation of terms, substitution (of numbers for letters), addition, and so on, without any attempt to show algebra as generalised arithmetic, and, for many pages, with no indication that all this "symbol-juggling" has any relation to practical problems. The subject is developed as far as simple equations in one unknown, and then concludes with an excellent set of exercises on substitution in, and changing the subject of, "real" formulae, drawn from many branches of science and engineering, the use of each formula, and the meaning of the symbols, being clearly given.

A chapter on logarithms follows. The decimal logarithms are introduced rather suddenly by the statement (following some instances of integral indices and logarithms) "Since $2 = 10^{\cdot 3010}$, $\log 2 = \cdot 3010$,"

but the student is vouchsafed no information as to how such a fact is known, and is not referred to the tables at the end of the book for several pages. The exercises in this section are purely mechanical—exercises in computation.

Then comes a section upon mensuration, confined to areas of rectilineal figures and the circle, and to volumes of prisms and cylinders. The final chapter gives briefly the principle, and rules for the use, of the slide-rule.

What strikes one most, especially when remembering the tendencies of modern school text-books, is the lack of any consistent attempt to interest the student, or to make him think for himself. Those, however, who have had experience of the grade of student for which this book is intended will probably agree with—and deplore—the statement that they will *do* anything sufficiently resembling the model they have been shown, but that to expect from them even a little original thinking is to be doomed to disappointment. As things unfortunately are, then, and until they improve, this book may possibly be regarded as satisfactory by many teachers of junior evening technical students—but as long as it is so regarded, improvement will be delayed. W. G. B.

Introduction to the Theory of Fourier's Series and Integrals. By H. S. Carslaw. Pp. xiii + 368. 20s. 1930. (Macmillan.)

We know from earlier editions that this book is very much more than its title claims it to be. Besides presenting the theory of Fourier's series and integrals, it gives an exact and admirably clear account of the necessary ideas of pure analysis—such as continuity, integration and the general theory of infinite series and limiting processes. Indeed only a third of the book is taken up with trigonometrical series and only a dozen pages with trigonometrical integrals.

At each successive edition the author has to decide on what basis of pure analysis his treatment of Fourier's series shall rest. His aim is to expound the properties of Fourier's series arising from the " ordinary " functions which appear in the mathematical treatment of any science. He is naturally reluctant to develop advanced technique designed to deal with less ordinary functions which interest only the pure mathematician. Against this Professor Carslaw has to balance the fact that the broader the basis of analysis is made the easier it is to build on it the theory of Fourier's series. The theorems take a sharper form and are proved by more direct arguments ; exceptional cases and restrictions disappear when a more general point of view is taken.

Comparing the present edition with the second, we are glad to see inserted in the early chapters paragraphs dealing with upper and lower limits and with functions of bounded variation. An addition of much greater size and importance is an account of Lebesgue's theory of integration. This, however, forms an appendix and the author still bases his discussion of Fourier's series on the older (Riemann) integral, referring the reader elsewhere for the more comprehensive treatment based on the Lebesgue integral.

A student might reasonably think that to understand this more comprehensive theory of Fourier's series he must serve a long apprenticeship to the theory of sets of points and real functions. Professor Carslaw could have encouraged him by emphasizing that the important thing is the *existence* of an integral which has simple and convenient properties. The integral is a machine which will work in the hands of those who are not interested in its design and construction.

Since the first edition of this book, Professor Carslaw has doubled its value. Our only regret is that he defies custom by not halving the first coefficient of a Fourier series. J. C. Burkill.

A Simplified Presentation of Einstein's Unified Field Equations. By T. Levi-Civita. Translated by J. Dougall. Pp. 22. 2s. 1929. (Blackie.)

For a long time Einstein has been engaged upon an attempt to formulate a Unified Field Theory from which he hopes (in spite of the scepticism of his colleagues) to derive all the laws of physics, electrical as well as gravitational. For this purpose he finds it necessary to attribute to every element of space-time something in the nature of a direction, so that it is possible to speak of " parallel " directions at different points (*teleparallelism*) with some but not all of the Euclidean implications of the word. Einstein's first papers on this subject were very complicated, and Levi-Civita's form is certainly a simplifica-

tion, though to many it is itself sufficiently mysterious, especially if they are not acquainted with the author's book on *Absolute Differential Calculus*.

Recently Einstein has given a new version of his theory, abandoning some features of his former treatment. He acknowledges his indebtedness to the French geometer Cartan (whose researches on the curvature and torsion of space are explained by Professor Whittaker in the *Journal* of the London Mathematical Society, Vol. 5, Part 1). In reply to a question at the close of a lecture at Nottingham, Einstein stated that this new version is not compatible with Levi-Civita's work. H. T. H. PIAGGIO.

Intermediate Mechanics: Dynamics. By D. HUMPHREY. Longmans' Modern Mathematical Series. Pp. xii + 382. 10s. 6d. 1930. (Longmans, Green.)

This course of Mechanics is intended for the upper forms of schools, and the first volume, which is now published, deals with Dynamics. The methods of the Calculus are used throughout, and there is a large collection of examples drawn from various Higher Certificate and University Scholarship examinations. It is unfortunate that in other respects the book retains the disadvantages of some of the older text-books on the subject. We give a few examples.

In Chapter I we have, "A point or body may have several different velocities simultaneously, *e.g.* a person walking on the deck of a ship in motion". No attempt is made to analyse this statement, which presents a real difficulty, not only to the beginner. It is unnecessary to introduce the artificial idea of a body having two velocities when what we mean is that the body has one velocity relative to a given frame of reference which has a velocity relative to another frame. There is again no indication of the frames of reference which are permissible in the discussion of Newton's laws of motion. There is no detailed discussion of force and mass, and the fundamental equation is given in the form $P = mf$.

The difficulty of distinguishing between mass and weight need not arise if a full discussion is given in either of the ways suggested in the recent Mathematical Association Report on *The Teaching of Mechanics in Schools*. The definition, "The Mass of a body is the quantity of matter in the body" is by itself unconvincing to the student; the idea can be made clearer by distinguishing at first between "gravitational mass" and "inertial mass", obtained from collision phenomena.

The chapter on collisions makes no new departure. It would be valuable here to include an elementary treatment of the collision of two railway carriages with buffers such as is considered in Poynting and Thomson's *Properties of Matter*. This can be illustrated by a simple experiment as described in the Report.

The last two chapters are devoted to the two-dimensional Dynamics of a Rigid Body and include useful sections on Moments of Inertia and the Instantaneous Centre. The significance of d'Alembert's Principle would be made clearer by connecting it with the Principle of Virtual Work. The internal forces in a rigid body are then considered as forces of constraint and do no work in displacements, such as translation or rotation of the body as a whole, which are consistent with the constraints. It is definitely misleading to speak of the "molecular reactions of the neighbouring particles of the body".

The book is well printed and the diagrams clear, although a notation for distinguishing between the forces producing motion and the mass-accelerations would be an improvement. B. SWIRLES.

(1) **Exercises in Algebra for Beginners.** By W. S. BEARD. Pp. 64. 10d.; answers, 6d. 1929. (Bell.)

(2) (i) **Junior Algebra Test Papers**; (ii) **Senior Algebra Test Papers.** By W. G. BORCHARDT. Pp. 64 + xvi, 88 + xvi. 1s. 3d. each. 1929, 1930. (Rivingtons.)

(3) **Graph Book.** By C. V. DURELL and A. W. SIDDONS. Pp. 80. 1s. 9d. or in stiff boards, 2s.; with answers, etc., 2s. 6d.). 1929. (Bell.)

(1) This book provides a first year's course in Algebra. The author claims that it can be taken as a "direct stepping-stone" to Mr. Durell's *Practical School Algebra*, and the claim is justified. Nothing further need be said in commendation of the book. There is sufficient explanatory text to show the scheme on which the book has been planned.

(2) Each of these little books has in it fifty papers of six questions each. The Senior Papers are a continuation of the previously published Junior Papers, and the whole series is graduated, ending with questions on the Calculus. The questions are well chosen and interesting. Books to be recommended.

(3) An Exercise Book which is also a Text Book. It covers all the ground—even more than all the ground—of the School Certificate Examinations. The arrangement provides that working should be adjacent to text; but at the end there are several spare pages of squared paper. This also is a book to be recommended. T. M. A. COOPER.

Fundamental Arithmetic Cards. Standard I. Sets A and B. By P. B. BALLARD. 1s. 3d. per set. 1929. (Univ. of London Press.)

These cards are for use by children of seven years of age, for whom they are more suitable than books. In each set there are twenty-four cards, and on each card is a well-printed set of graduated exercises. The cards are to be used in order, and a set of cards contains some 550 examples. Sets A and B are alternative, so that if a child wants more practice in the sums on A 6, for instance, he can work at B 6 before going on to A 7. The exercises are very well chosen, and a teacher with a class of *beginners* in Arithmetic can be safely advised to give these cards a trial. They form a first set of written exercises for Dr. Ballard's *Fundamental Arithmetic*.

T. M. A. COOPER.

Solid Analytical Geometry and Determinants. By A. DRESDEN. Pp. x+310. 15s. 1930. (Wiley and Chapman & Hall.)

This is a course for students in American colleges and universities. It begins with chapters on Determinants and Systems of Linear Equations, and these include much that is useful in the main body of the work. Another piece of preliminary work that would have been useful is an introduction to the elementary properties of Vectors, including at least the scalar product, for this would have enabled the author to economise space in the presentation of much of the later work. As so great a point is made at the start of matrices, it seems strange that the straight line is not habitually referred to in the form

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{vmatrix},$$

which is specially convenient to the student who is accustomed to use envelope or locus methods according to the nature of the problem in hand. In this book, however, little attention is paid to the reciprocal outlook or to the Plücker coordinates of a straight line. In fact the standpoint seems rather a narrow one. Elements at infinity and complex elements are considered as non-existent; this is a perfectly logical beginning and one with which the present reviewer is in full sympathy. It is a procedure which must, or at least ought to, be adopted in the corresponding introduction to two-dimensional geometry. But as the beginner in three-dimensional geometry will be already aware of these ideas, it does not seem satisfactory to make the course independent of them.

These criticisms are prompted perhaps by the needs of the English student and might have less force for the American students who use the book.

One more general matter may be mentioned: the book reads rather as if it had been written by an algebraist than by a geometer; for example, we are told that a vertex of a quadric is a point where $Q=Q_1=Q_2=Q_3=0$. And one more detail: it is impossible to quarrel with American spelling in an American book, but parallelopiped is presumably as incorrect in one language as in the other. A. R.

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F. L. GRANT and A. M. HILL
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R. WALKER Essentials of Arithmetic - - - - - - 1929
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From Mr. **H. G. Forder**:

K. REIDEMEISTER Grundlagen der Geometrie Springer's Grundlehren 32 1930

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Presentation copies from the author, for whom the donor had carried out a mass of the numerical calculations involved.

A. DE MORGAN Arithmetic {3} - - - - - - - - - 1835

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The Elements of Euclid {4} - - - - - - 1731
"With the uses of each proposition in all the parts of the mathematicks." Translated from French by R. Williams.

J. HARRIS The Circle and Straight-Line. I {2} - - - - 1876
A demonstration that the value of π is $20\sqrt{2}/9$.

R. HEGER Analytische Geometrie auf der Kugel Schubert 54 1908

D. HILBERT Foundations of Geometry - - - - - - 1902

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E. DE JONQUIÈRES Mélanges de Géométrie Pure - - - - - 1856
Includes a version of Maclaurin's *De Linearum Geometricarum Proprietatibus generalibus.*

N. L. DE LA CAILLE
Leçons élémentaires de la Mécanique { } - - 1764

J. LESLIE Rudiments of Plane Geometry - - - - - 1828

E. MACH Space and Geometry - - - - - - - 1906

F. C. M. MARIE Géométrie Stéréographique - - - - - - 1835
This is the reproduction described under J. L. Cowley's name in the Second Library List.

I. NEWTON Philosophiae Naturalis Principia Mathematica {T. Le Seur et F. Jacquier, rep.} 2-4 - - - - - - 1822
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The Mathematical Principles of Natural Philosophy (3 vols.) - - - - - - - - - - 1803
Motte's translation, revised by W. Davis.

K. PEARSON, A. LEE, and L. BRAMLEY-MOORE
Genetic selection: Inheritance of fertility - - 1899

J. PLAYFAIR Geometry {10: P. Kelland} - - - - - - 1846

T. S. ROW Geometric Exercises in Paper Folding - - - 1901
The first American edition.

J. W. WITHERS Euclid's Parallel Postulate - - - - - - 1905

From Sir **Thomas Muir,** a copy of his latest work:

Contributions to the History of Determinants, 1900-1920 - - - - - - - - - - 1930
A continuation of his four-volume treatise.

From Prof. **E. H. Neville,** a school book by C. V. Durell and A. Robson, together with:

P. DE LA HIRE New Elements of Conick Sections {2} - - - 1723
Translated from French by B. Robinson.
The first edition is in List 3.

J. OZANAM Cursus Mathematicus; V - - - - - - 1712
A volume of the English translation by J. T. Desaguliers.
Vol. III is in the Greenstreet bequest; can any member add other volumes?

Also from Miss **R. Binden,** a collection of back numbers of the *Gazette*; and from Mr. **H. E. Piggott** and Mr. **A. Robson,** school books by H. G. Forder and by A. W. Siddons and R. T. Hughes.

The following has been bought:

Fragen der Elementargeometrie; I - - - - - - - - 1911
F. Enriques' valuable collection of essays was published in Italian in 1900. The German translation was issued in two parts, the second, by H. Fleischer, in 1907, the first, by H. Thieme, four years later.
Vol. 2 has been an odd volume in the Library for many years.

CORRESPONDENCE.

To the Editor of the *Mathematical Gazette*.

SIR,—The very interesting and helpful Report on *The Teaching of Mechanics in Schools*, recently issued by the Mathematical Association, is, unfortunately, disappointing in the sections which deal with mass and weight. There will always be confusion, not only in the minds of beginners but even I imagine in the practice of engineers, so long as the two senses of the word "pound" are admitted. The committee says that it cannot be helped, but surely it is the duty of the teacher to present ideas clearly and give the budding mechanic a fair start.

It seems essential that the word "pound" should never in any circumstances be used in the sense of weight. A pound is a lump of matter whose weight is a poundweight (lbwt., *not* lb.wt.). This is the practical unit of the British engineer, crude and unscientific and lacking in constancy like the nautical mile, but eminently useful. The "absolute" unit, the "poundal" (lbl.), can only be introduced with dynamics and the idea of mass, and is lb. $\times$ ft. $\times$ sec.$^{-2}$.

It is essential also, in order to avoid the common confusion, that "g" should be used invariably for an acceleration, not a number. To say that the weight of a pound is g poundals is confusion worse confounded. After mass has been explained, it is correct to say that a mass m has a weight mg, but certainly not mg poundals; we may also say that a mass m lb. has a weight approx. $32m$ lbl., or a mass m gm. a weight approx. $980m$ dynes, but not mg dynes.

It is very desirable also that the term "foot-pound" of the engineer should be replaced by "foot-poundweight", or at least that the contraction ft.lbwt. should invariably be used, and never "ft.lb.", for the unit of work; for "ft.lb." is not work but distance $\times$ mass. If the phrase "foot-pound" is considered too deep-rooted to be displaced, at least we may contract it to ft.-lb. and not ft.lb. which means ft. $\times$ lb. The same objection applies to the new term "second-pound" for unit of impulse, which should be sec.lbwt. (not sec.lb.wt.).

To the purist also the term "horse-power" is commonly misused. Horse-power is the unit of power. It is really just as grotesque to say "the horse-power of an engine is 7" instead of "the power of an engine is 7 HP" as it would be to say "the pound-weight of a load is 20" instead of "the weight of a load is 20 lbwt.".

Are these not matters in which a teacher can exercise a reforming influence?

Wellington, N.Z., 15th Sept. 1930. D. M. Y. SOMMERVILLE.

Professor Sommerville's plea that the word "pound" should be used in one sense only is at first sight the only logical method of attempting to get rid of an admitted source of confusion.

But on careful consideration (and I can assure Professor Sommerville that the M.A. Committee gave this and the connected questions the most lengthy, careful and I may add at times heated consideration) this seemingly obvious course appears to leave out of account human nature and the way in which language grows up.

It is usage which settles the meanings of words and usage is against the proposal. It is human nature to resent dictation as to ordinary modes of speech from an academic body, and committees who wish their reports to influence teaching must try to avoid raising resentment.

After all, words used in more than one sense are fairly common in Mathematics and extraordinarily so in non-mathematical English.

What would be thought of a report on Geometry which said "The word

pole must never be used by itself; it must always be 'pole of a certain line with respect to a certain conic', and this is the only possible use of the word *pole*"; or of a treatise on English grammar which said "It is legitimate to speak of a *row* of houses or a *row* of men, and the word *row* must on no account be used in any other sense"?

These instances may indicate some of the reasons why in this question the Committee decided that "the better part of valour is discretion".

C. O. TUCKEY.

THE PILLORY.

"In areal co-ordinates, if the equation $(lx)^{1/2}+(my)^{1/2}+(nz)^{1/2}=0$ represents a central conic, prove that the centre is $(m+n, n+l, l+m)$.

If, however, the equation represents a parabola, prove that $l+m+n=0$ *and that the line* $lx+my+nz=0$ *is parallel to the axis.*"—Examination for Entrance Scholarships, St. Catherine's College, Cambridge, Jan. 1929.

[If $(m+n, n+l, l+m)$ is on the line at infinity, whose equation is

$$x+y+z=0,$$

then $l+m+n=0$; the coordinates can be replaced by (l, m, n), and the point (l, m, n) does not lie on the suggested line unless $l^2+m^2+n^2=0$! The simplest line through (l, m, n) is

$$(m-n)x+(n-l)y+(l-m)z=0,$$

but this is a diameter in every case, not merely in the case of the parabola.

If the conic is a parabola, the focus is $(a^2/l, b^2/m, c^2/n)$, and the equation of the axis is, actually,

$$\frac{x}{l}\left(\frac{b^2}{m^2}-\frac{c^2}{n^2}\right)+\frac{y}{m}\left(\frac{c^2}{n^2}-\frac{a^2}{l^2}\right)+\frac{z}{n}\left(\frac{a^2}{l^2}-\frac{b^2}{m^2}\right)=0. \qquad \text{H. E. P.]}$$

AN EXPLANATION.

MRS. GREENSTREET asks us to say, with reference to a phrase in the October *Gazette* that has been misunderstood, that she was appointed to the Marling School as matron immediately on leaving the hospital in which she qualified. She held no other position at Stroud, nor was she attached to the headmaster's family at any time in any professional capacity whatever. Dr. Macaulay explains that what he wished to recall was that Miss Teal, as she then was, was always addressed both in the school and in the family by the warm appreciative title of nurse, not by the emotionless title which was her official due.

A MEMENTO OF MR. GREENSTREET.

The articles on Mr. Greenstreet which occupied the first six pages of the October *Gazette* have been reprinted with a photograph; this memento is to be obtained from Mr. Pendlebury at the price of 6d., postage included.

AN OFFER.

The Association has for disposal several runs of the *Gazette*, vols. 3-10, complete in parts as issued except for the Title-page and Index to vol. 3. These are offered at six guineas a set, carriage paid (in U.K.). It must be understood that the missing Title-page and Index can not be supplied separately at any price, and that vols. 1-2 are not available to supplement these sets.

Applications should be made to the Librarian, who will also answer enquiries as to complete sets.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
AT THE UNIVERSITY PRESS, GLASGOW.